Spread rates of spread models with frozen symbols

Jung-Chao Ban[1], Jyy-I Hong*[1], and Yu-Liang Wu[2]

[1]Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan, ROC.
[2]Department of Mathematical Sciences, P.O. Box 3000, 90014 University of Oulu, Finland

Abstract

This article aims to compare the long-term behavior of a spread model before and after a type in the model becomes frozen; namely, a type of which an individual only produces individuals of the same type. By means of substitution dynamical systems and matrix analysis, the first result of this work gives the spread rates of a 1-spread model with one frozen symbol. Later in the work, this is shown to hold under more general settings, which include generalized frozen symbols and frozen symbols in m-spread models. Numerical experiments are provided as supporting evidence for the theory.

Modelling always plays a significant role in disease control, prevention and decision making. We propose a mathematical model to study the population in which certain type of individuals will be blocked or removed. Using these models, we are able to consider the influence on the current population by its history within certain time period, predict the long-term behavior of the spread rate, and describe the transition of the spread pattern during the pandemic period. Our methodology gives a comparison between two models which can be used as a reference to determine the efficiency of the policies for preventing the pandemic outbreak. Some numerical experiments with different irreducible structures are also provided to show the spreading tendency as well as the exponential decay when a type is frozen to support the theoretical results.

*Corresponding author. *Email address*: hongjyyi@nccu.edu.tw
Keywords: spread model, spread rate, frozen symbol.

1 Introduction

1.1 Motivations

A population is often divided into several groups for some purposes. For example, according to WHO COVID-19 Case definition [10], patients fulfilling the definition can be categorized as suspected, probable or confirmed case for surveillance and epidemic investigation purposes. People in different categories may require different treatments and have different impacts on the disease control. In this paper, we consider a population in which individuals are categorized into groups according the patterns how they spread certain virus. Individuals in the same category share the same spread pattern and are said to be of the same type. Especially, in this work, we assume that the population consists of individuals of K different types, say a_1, a_2, $\cdots$, a_K.

The purpose of this paper is to introduce a spread model to describe the spread of a biological system and gives a full characterization on the spread rate of all types in such a system. As every now and then the disease outbreaks greatly affect many aspect of life, experts in different fields are striving to propose mathematical models that either explain or predict the trend of the pandemic in order to aid decision making and minimize the impact (cf. [4, 6, 15, 9, 7, 11, 5, 1, 14, 13]). Among these research topics, one that attracts much of attention is the characterization of the growth of the number of the infected/deceased when certain measures, such as mandatory wearing mask or quarantine, are taken to prevent the spread. For instance, in [16], the authors extend their earlier work to apply the logistic map model to the COVID-19 and explain the decrease in fatality rate in terms of the decrease of parameter d_∞ after the vaccination steps in. This is also the main idea of this work. Based on the previous works of the authors [2, 3], in which two kinds (topological and random) of spread models are discussed, this article continues to explore the long-term behavior of the spreading after the transmission is blocked by means of the aforementioned measures, or in terms of the spread model, a type f (or more generally, a set $F = \{f_i\}_{i=1}^R$ as defined in Section 3.4) is 'frozen' so that any individual of this type fails to produce any child other than exactly one of type f. As shown later, the number of type-f individuals are significantly reduced in the long run after f is frozen. In the following paragraph, we introduce the spread model and spread rate more precisely.

1.2 1-spread models and spread rate

Let $\mathcal{A} = \{a_i\}_{i=1}^K$ be a type set, and $\mathcal{T}_d$ be the conventional d-tree for $d \in \mathbb{N}$ with the root ϵ. Define $\Sigma^s = \{g \in \mathcal{T}_d : |g| = s\}$ for $s \in \mathbb{N}$ and $\Delta_n^h = \{g \in \mathcal{T}_d : g \text{ is a descendant of } h \text{ with } |g-h| < n\}$, where $|g-h|$ represents the length of the unique path from h to g and $|g| = |g-\epsilon|$. If $h = \epsilon$, we simply write $\Delta_n^\epsilon = \Delta_n = \cup_{i=0}^{n-1}\Sigma^i$.

For convenience, we write

$$\Delta_m^n = \Delta_n \backslash \Delta_m = \{g \in \mathcal{T}_d : m \leq |g| < n\},$$

and for $F \subseteq \mathcal{T}_d$ we define $F_m^n = F \cap \Delta_m^n$. For a finite set $F \subseteq \Delta_1$, a function $p : F \to \mathcal{A}$ is called a 1-*pattern* and $F = F_p$ is called the *support* of p. Denote by $\mathcal{P}_1$ the set of all 1-patterns. For $p \in \mathcal{P}_1$, write $p^{(0)} = p(\epsilon) \in \mathcal{A}$ and for $g_1, \ldots, g_{d_p} \in F_p$ with $|g| = 1$, $d_p \in \mathbb{N}$, we write $p^{(1)} = (p(g_1), p(g_2), \ldots, p(g_{d_p}))$. Thus, the 1-pattern (cf. Figure 1) can also be written as

$$\begin{aligned} p &= (p^{(0)}; p^{(1)}) \qquad (1) \\ &= (p(\epsilon); p(g_1), p(g_2), \ldots, p(g_{d_p^{(1)}})). \end{aligned}$$

Let $\mathcal{S} = \{p_i\}_{i=1}^L \subseteq \mathcal{P}_1$ and set $d = \max_{p\in\mathcal{S}} d_p$. The corresponding d-tree $\mathcal{T}_d$ is defined in the preceding paragraph, and other notations, e.g., Δ_m^n, F_m^n, etc., are also defined. The set $\mathcal{S}$ is called a *spread model* if for every $g \in F_p$ with $|g| = 1$, there exists a unique $q \in \mathcal{S}$ such that $q^{(0)} = p(g)$. Given a 1-spread model $\mathcal{S}$ and $p \in \mathcal{S}$, we define τ_p^∞ as follows. Let $\tau_p^0 = p^{(0)}$ and $\tau_p^1 = p$. For $g \in F_p$ with $|g| = 1$, since $\mathcal{S}$ is a 1-spread model, there exists an $q_g \in \mathcal{S}$ with $q_g^{(0)} = q_g(\epsilon) = p(g)$. Thus, we replace $p(g)$ by the 1-pattern q_g for all $g \in F_p$ with $|g| = 1$ to generate a pattern τ_p^2 (cf. Figure 2). Once τ_p^n is constructed, we substitute the pattern q_g for the symbol $\tau_p^n(g)$, $g \in F_{\tau_p^n}$ with $|g| = n$, to generate τ_p^{n+1}. Finally, we define $\tau_p^\infty = \lim_{n\to\infty} \tau_p^n$ (write $\tau_p = \tau_p^\infty$ to shorten the notation) and call it the *infinite spread pattern induced from p with respect to S* (or *induced spread pattern from* p, see Figure 3). Let τ_p for some $p \in \mathcal{S}$ and $p^{(0)} = p(\epsilon) = b$. Suppose $F \subseteq F_{\tau_p}$ is a finite set, we denote by $\tau_p|_F$

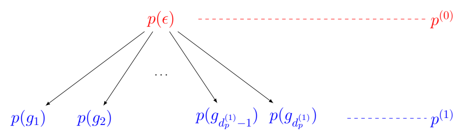


Figure 1: Illustration of a 1-pattern

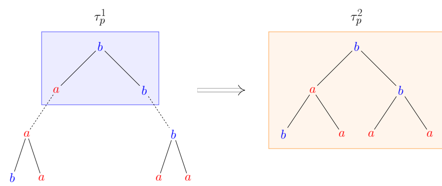


Figure 2: Generation of τ_p^2

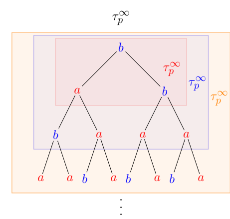


Figure 3: Generation of τ_p^∞

the sub-pattern of τ_p along the subset F; that is, $\tau_p|_F = \{\tau_p(g) : g \in F\}$. Let $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$ and $s_n = \sum_{i=1}^n k_i$. The following value $s_b(a; \{k_n\}_{n=1}^\infty)$ is interesting and important for the spread model.

$$s_b(a; \{k_n\}_{n=1}^\infty) := \lim_{n\to\infty} s_b(a; [s_n, s_{n+1}]) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|}, a \in \mathcal{A}, \tag{2}$$

where $O_a(\tau_p|_F)$ denotes the number of occurrences of type a in the range F, and $|F|$ is the size of the set F. Obviously, the value $s_b(a; \{k_n\}_{n=1}^\infty)$ indicates the average of type a spread over the range $\Delta_{s_n}^{s_{n+1}}$, as $n \to \infty$, with the initial pattern $p \in \mathcal{S}$ (or initial type $b \in \mathcal{A}$). More precisely, since the range $\Delta_{s_n}^{s_{n+1}}$ represents the set of lattices in the s_nth, $(s_n + 1)$th, $\cdots$, $(s_{n+1} - 1)$th levels, the spread rate $s_b(a; \{k_n\}_{n=1}^\infty)$ means the proportion of the individuals of type a in these levels in the population initiated with an individual of type b in the long run. Using the theory of substitution dynamical systems, $s_b(a; \{k_n\}_{n=1}^\infty)$ can be calculated as the ath component of the eigenvector of the associated ξ-matrix (defined later), where ξ is a substitution induced by the spread model. In [2], the random spread model is also built, and the theory of computation of $s_b(a; \{k_n\}_{n=1}^\infty)$ is also addressed.

1.3 1-spread model with a frozen symbol and spread rate

An interesting question arises naturally.

Problem 1. *Given a 1-spread model $\mathcal{S}$, if a certain type $f \in \mathcal{A}$ (or a set $F = \{f_i\}_{i=1}^R \subseteq \mathcal{A}$) has become non-infectious, e.g., f has been quarantined or recovered, how does the new type f (or the new* 1*-spread model $\mathcal{S}^f$) affect the spread rate (2) of some $a \in \mathcal{A}$?*

As we mentioned before, Problem 1 is important since if we could find such $f \in \mathcal{A}$ in which the spread rate decreases significantly, further actions, such as isolating f or healing f, could be accomplished. To answer Problem 1, we define the 1-*spread model with a frozen symbol*[1] $f \in \mathcal{A}$ as follows. Let $\mathcal{S} = \{p_i\}_{i=1}^L$ be a 1-spread model and $f \in \mathcal{A} = \{a_i\}_{i=1}^K$ be a *frozen symbol*. For $p_f \in \mathcal{S}$ with $p_f^{(0)} = f$ (note that such p_f is unique since $\mathcal{S}$ is a spread model), we replace the pattern p_f with

$$\bar{p}_f = (\bar{p}_f^{(0)}; \bar{p}_f^{(1)}) = (f; f). \tag{3}$$

Define $\mathcal{S}^f = (\mathcal{S} \backslash \{p_f\}) \cup \{\bar{p}_f\}$ and call it a *1-spread model with frozen symbol* f. Let $\mathcal{S}^f$ be defined as above and the induced spread pattern from p, say τ_p^f, be given. Given $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$, we set $s_n = \sum_{n=1}^n k_n$. The objective of this investigation is to calculate the spread rate $s_p^f(a; \{k_n\}_{n=1}^\infty)$ (defined as (2)) of $a \in \mathcal{A}$ with respect to $\mathcal{S}^f$.

Let $\mathbf{M} = \mathbf{M}_f$ be the associated ξ-matrix of $\mathcal{S}^f$ (defined in Section 2). As $\mathbf{M}$ is no longer irreducible, it can be of the form (4) and (24). Let $r \in \mathbb{N}$ be the number of irreducible components of M. Proposition 2 presents the theory for the calculation of the spread rate $s_b(a; \{k_n\}_{n=1}^\infty)$ when k_n is a single layer. Theorem 8 extends Proposition 2 to the case where $r = 2$, and the general cases of $r > 2$ are handled by Theorem 14. The spread rate for the 1-spread model with a frozen set is discussed in Section 3.4. Section 4 focuses on the spread rate of an $\mathcal{S}^f$ within a constant or increasing range. More precisely, Theorem 18 addresses the cases where $s_n = \sum_{i=1}^n k_i$ is an unbounded sequence of integers. Finally, the spread rate for an m-spread model is examined in Section 5, and some numerical results are presented in Section 6.

[1] We call such symbol *frozen* is because $\bar{p}^{(0)} = \bar{p}^{(1)} = f$, it forces $\tau_{\bar{p}}(g) = f$ for all $g \in F_{\tau_{\bar{p}}}$ and $F_{\tau_{\bar{p}}}$ is just an infinite 1-d single chain. This means $\tau_{\bar{p}}$ cannot spread, and therefore be 'frozen'.

2 Preliminaries

2.1 Substitutions, ξ-matrix and spread rate

Let $\mathcal{A} = \{a_i\}_{i=1}^K$ be a type set, and set $\mathcal{A}^* = \cup_{m=0}^{\infty} \mathcal{A}^m$, i.e., the monoid of $\mathcal{A}$, where $\mathcal{A}^m$ is the set of blocks of $\mathcal{A}$ of length $m \in \mathbb{N}$. Suppose $\mathcal{S}$ is a 1-spread model and $p = (p^{(0)}; p^{(1)}) = (b; p^{(1)}) \in \mathcal{S}$. The associated *substitution* ξ on $\mathcal{A}$ is a map $\xi : \mathcal{A} \to \mathcal{A}^*$ defined by $\xi(b) = p^{(1)}$, where the word length $|\xi(b)|$ of $\xi(b)$ equals $d_p^{(1)}$, the number of $g \in F_p$ with $|g| = 1$. This substitution induces a morphism of $\mathcal{A}^*$ by putting $\xi(B) = \xi(b_0)\xi(b_1)\cdots\xi(b_n)$ if $B = b_0 b_1 \cdots b_n \in \mathcal{A}^*$ and $\xi(B) = \emptyset$ if $B = \emptyset$. Denote by $\xi^n = \xi \circ \xi^{n-1}$ the *n-time iteration map* of ξ. Suppose ξ is a substitution of a spread model $\mathcal{S}$. The associated *ξ-matrix* $M = M_\xi$, which is a $K \times K$ matrix with each entry in $\{0, 1\}$, is defined by

$$M = [m_{ij}] := [O_{a_i}(\xi(a_j))],$$

recalling that $O_{a_i}(\xi(a_j))$ is the number of occurrences of the type a_i in the pattern $\xi(a_j)$. Furthermore, if $\omega \in \mathcal{A}^*$ and $L(\omega) \in \mathbb{R}^K$ denotes the vector whose components are $O_{a_i}(\omega)$ for $1 \le i \le K$, it is clear that $L(\xi(\omega)) = M \cdot L(\omega)$ and that

$$L(\xi^n(a_j)) = M^n L(a_j),$$

where $L(a_j) = [0, \ldots, 0, \overbrace{1}^{j\text{th}}, 0, \ldots, 0]$ and $|\xi^n(a_j)| = \mathbf{1}^K M^n L(a_j)$, $\mathbf{1}^K \in \mathbb{R}^K$ with all entries are all $1'$s. For more details we refer the reader to [12].

Theorem 3.1 [2] provides a method to compute $s_b(a) := s_b(a; \{k_n\}_{n=1}^\infty)$ with respect to $\{k_n\}_{n=1}^\infty$ with $k_n = 1$ $\forall n \ge 1$. That is, for all $p \in \mathcal{S}$ with $p^{(0)} = b$, we have

$$s_b(a) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|} = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Sigma^{s_n}(\tau_p)})}{|\Sigma^{s_n}(\tau_p)|} = v(a),$$

where $v(a)$ is the ath component of the right positive eigenvector v of the ξ-matrix M corresponding to the maximal eigenvalue ρ_M of M.

2.2 Irreducible components

Let $f \in \mathcal{A}$ be a frozen symbol and $\mathcal{S}^f$ be the 1-spread model with frozen symbol f. We continue in the same fashion as above to define the substitution ξ^f, and set $\mathbf{M}_0 = M_\xi$ and $\mathbf{M} = M_{\xi^f}$ the associated ξ-matrix and ξ^f-matrix respectively. Since $\mathbf{M}_0$ is a $K \times K$ binary matrix, we denote by $G_{\mathbf{M}_0} = (V_{\mathbf{M}_0}, E_{\mathbf{M}_0})$ the associated graph, that is $V_{\mathbf{M}_0} = \mathcal{A}$ and $E_{\mathbf{M}_0} = \{(a_i, a_j) \in V_{\mathbf{M}_0} \times V_{\mathbf{M}_0} : \mathbf{M}_0(a_i, a_j) = 1\}$. It is clear that $\mathbf{M}$ is obtained from $\mathbf{M}_0$ by deleting all edges of the form $(f, a) \in E_{\mathbf{M}_0}$. Therefore, we have

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix}, \tag{4}$$

where $\mathbf{C} \in \mathbb{R}^{1\times(K-1)}$ and $M \in \mathbb{R}^{(K-1)\times(K-1)}$. We emphasize that two additional assumptions about the complexity of the spreading are made as in [2]; namely, $\lim_{n\to\infty} |\xi^n(\alpha)| = \infty \ \forall \alpha \in \mathcal{A}$ and there exists $\xi(\alpha_0)$ beginning with α_0. These two conditions ensure that the associated ξ-matrix $\mathbf{M}_0$ is primitive, i.e., $(\mathbf{M}_0)^k > 0$ for some $k \in \mathbb{N}$. Nevertheless, the matrix M defined in (4) is no longer necessarily primitive. Hence, we suppose that

$$M = \begin{bmatrix} A^{[11]} & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ A^{[r1]} & \cdots & \cdots & A^{[rr]} \end{bmatrix} \tag{5}$$

is the irreducible decomposition of M. As mentioned above, for irreducible $A \in M_{n\times n}(\mathbb{R})$, we denote by $G_A = (V_A, E_A)$ the associated graph, where V_A, E_A are the vertex and edge set of the graph G_A, respectively. The best general reference here is [8]. Here and subsequently, we use $a \in V_A$ to as the symbol a in the index set of the matrix A. Surprisingly, the rigorous value of the spread rate $s_b^f(a; \{k_n\}_{n=1}^\infty)$ is highly dependent on the choice of the 'initial type' b and 'target type' a. In the next section, we divide the discussion into cases where $r = 1$, $r = 2$ and $r > 2$.

Let $\mathcal{M}_{m\times n}$ be the set of $m \times n$ binary matrices. In what follows, if $A \in \mathcal{M}_{K\times K}$ admits exactly one eigenvalue, say ρ_A, satisfying $\rho_A > \lambda$ for all eigenvalue λ of A other than ρ_A, we call ρ_A the *maximal eigenvalue* of A. Meanwhile, the corresponding eigenvector, say v_A (resp. w_A), is called the *right* (resp. *left*) *maximal eigenvector* of A. To simplify the notation, we define $\rho = \rho_{\mathbf{M}}$, $\rho_0 = \rho_{\mathbf{M}_0}$, $\mathbf{v} = v_{\mathbf{M}}$, $\mathbf{v}_0 = v_{\mathbf{M}_0}$, $\mathbf{w} = w_{\mathbf{M}}$ and $\mathbf{w}_0 = w_{\mathbf{M}_0}$, and $\bar{v}$ stands for the normalized vector of v. Finally, we set as $\mathbf{1}_n$ the vector in $\mathbb{R}^n$ with every entry

1 and

$$I_m^n = [0, \dots, 0, \overbrace{1}^{m\text{th}}, 0, \dots, 0]^t \in \mathbb{R}^n \tag{6}$$

for $n, m \in \mathbb{N}$ with $n \geq m$ and $I_m^K = e_m$.

3 1-spread model with a frozen symbol: single layer

Throughout the paper, for a matrix A we denote by $A_{(i)}$ (resp. $A^{(j)}$) the vector of the ith row (jth column) of A.

3.1 The case where $r = 1$

Proposition 2 ($r = 1$)**.** *Let $S = \{p_i\}_{i=1}^L$ be a spread model and $S^f = \{q_i\}_{i=1}^L$ be the spread model with a frozen symbol $f \in \mathcal{A} = \{a_i\}_{i=1}^K$. Suppose the associated ξ^f-matrix $\mathbf{M}$ is of the form (4), and M is primitive, then for all $b \in V_M$ we have*

$$s_b^f(a) = \bar{\mathbf{v}}(a) = \bar{v}_{\mathbf{M}}(a) > 0, \, \forall a \in V_M \cup \{f\}.$$

Furthermore, $s_f^f(a) = 0 \; \forall a \in V_M$ and $s_f^f(f) = 1$.

Remark 3. *Proposition 2 tells that if the population is initiated with an individual of non-frozen type b and the ξ^f-matrix $\mathbf{M}$ is of the form*

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix},$$

with M primitive, then the group of the individuals of type a will survive and eventually the proportion of individuals of type a in the population will tend to the ath component of the normalized right eigenvector associated with the maximal eigenvalue of the ξ^f-matrix $\mathbf{M}$. On the other hand, if the population is initiated with an individual of the frozen type f, then the whole population will only consist of one individual of the frozen type in the long run. The case when the matrix M is no longer primitive will be discussed in the next subsection.

Proof. **1**. Since M is primitive, we have $\rho_M > 1$ and $\rho = \rho_M > 1$. Suppose $a, b = p^{(0)} \in V_M$, say $b = a_i$ and $a = a_j$, where $1 \leq i, j \leq K - 1$ we have

$$s_b^f(a) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Sigma^n(\tau_p)})}{|\Sigma^n(\tau_p)|} = \lim_{n\to\infty} \frac{(\mathbf{M}^n e_i)(j)}{\mathbf{1}_K \mathbf{M}^n e_i} = \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\mathbf{1}_K \mathbf{M}^n e_i}. \tag{7}$$

Assume that $\mathbf{w} = [w, c_1]$ and $\mathbf{v} = [v, c_2]$, where $0 \neq c_i \in \mathbb{R}$ for $i = 1, 2$. First, we claim that

$$\lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\rho^n} = c_3 \mathbf{v}(j), \tag{8}$$

where $0 < c_3 \in \mathbb{R}$ is a constant. Indeed, if $\mathbf{M} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$ and $M = PDP^{-1}$ are the Jordan decomposition of $\mathbf{M}$ and M, respectively, it can be easily checked that

$$\mathbf{P} = \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix}, \mathbf{P}^{-1} = \begin{bmatrix} P^{-1} & 0 \\ R & 1 \end{bmatrix} \text{ and } \mathbf{D} = \begin{bmatrix} D & 0 \\ 0 & 1 \end{bmatrix},$$

where Q, $R \in \mathbb{R}^{1\times k}$. Assume $D_{11} = \rho_M = \rho$, we obtain that $\left(P^{-1}\right)_{(1)}^t = [w]^t \in \mathbb{R}^{K-1}$ (resp. $P^{(1)} = [v] \in \mathbb{R}^{K-1}$) is the left (resp. right) Perron eigenvector of ρ_M. Since M is primitive, we have $\left(P^{-1}\right)_{(1)} = [w]^t > 0$. Thus, $c_3 := \left(P^{-1} I_i^{K-1}\right)(1) > 0$ for $1 \le i \le K-1$ (recall that $\left(P^{-1} I_i\right)(j)$ is the jth component of $P^{-1} I_i$). Note that

$$\frac{e_j^t \mathbf{M}^n e_i}{\rho^n} = \frac{e_j^t \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} e_i}{\rho_M^n} = \frac{e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & 1 \end{bmatrix} e_i}{\rho_M^n}, \tag{9}$$

that we have $|\lambda| < \rho_M$ if λ is an eigenvalue of M with $\lambda \neq \rho$, and that ρ_M is simple (since M is primitive). It follows from (9), from $\mathbf{v} = [v, c_2]$ and from $1 \le j \le K-1$ that

$$\begin{aligned}
\lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\rho^n} &= \lim_{n\to\infty} \frac{e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & 1 \end{bmatrix} e_i}{\rho_M^n} \\
&= \lim_{n\to\infty} e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} \frac{D^n}{\rho_M^n} & 0 \\ 0 & \frac{1}{\rho_M^n} \end{bmatrix} \begin{bmatrix} P^{-1} I_i^{K-1} \\ R I_i^{K-1} \end{bmatrix} \\
&= e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} e_1^t \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} P^{-1} I_i^{K-1} \\ R I_i^{K-1} \end{bmatrix} \\
&= c_3 \times e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} e_1 \\
&= c_3 \times e_j^t \mathbf{P}^{(1)} \\
&= c_3 \times \mathbf{v}(j).
\end{aligned}$$

Thus, (8) follows. Next, we claim that

$$\lim_{n\to\infty} \frac{\mathbf{1}_K \mathbf{M}^n e_i}{\rho^n} = c_3 \sum_{i=1}^{K} \mathbf{v}(i). \tag{10}$$

Following the same argument as above, we also have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathbf{1}_K \mathbf{M}^n e_i}{\rho^n} &= \lim_{n\to\infty} \mathbf{1}_K \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} \frac{D^n}{\rho_M^n} & 0 \\ 0 & \frac{1}{\rho_M^n} \end{bmatrix} \begin{bmatrix} P^{-1} I_i^{K-1} \\ R I_i^{K-1} \end{bmatrix} \\
&= c_3 \times \mathbf{1}_K \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= c_3 \times \mathbf{1}_K \mathbf{P}^{(1)} = c_3 \sum_{i=1}^{K} \mathbf{v}(i). \qquad (11)
\end{aligned}$$

Combining (10), (8) with (7) yields

$$s_b^f(a) = \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\mathbf{1}_K \mathbf{M}^n e_i} = \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i / \rho^n}{\mathbf{1}_K \mathbf{M}^n e_i / \rho^n} = \frac{c_3 \mathbf{v}(j)}{c_3 \sum_{i=1}^{K} \mathbf{v}(i)} = \frac{\mathbf{v}(j)}{\sum_{i=1}^{k+1} \mathbf{v}(i)} = \bar{\mathbf{v}}(j),$$

which is the desired equality.

2. It can easily computed that

$$\begin{aligned}
s_f^f(a) &= \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_K}{\mathbf{1}_K \mathbf{M}^n e_K} \\
&= \lim_{n\to\infty} \frac{e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & 1 \end{bmatrix} e_K}{\mathbf{1}_K \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ R & 1 \end{bmatrix} e_K} \\
&= \lim_{n\to\infty} \frac{e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0_k \\ 1 \end{bmatrix}}{\mathbf{1}_K \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} D^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0_k \\ 1 \end{bmatrix}} \\
&= \lim_{n\to\infty} \frac{e_j^t \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} 0_k \\ 1 \end{bmatrix}}{\mathbf{1}_K \begin{bmatrix} P & 0 \\ Q & 1 \end{bmatrix} \begin{bmatrix} 0_k \\ 1 \end{bmatrix}} \\
&= \lim_{n\to\infty} \frac{e_j^t \begin{bmatrix} 0_k \\ 1 \end{bmatrix}}{\mathbf{1}_K \begin{bmatrix} 0_k \\ 1 \end{bmatrix}}.
\end{aligned}$$

Hence,

$$s_f^f(a) = \begin{cases} 0 & \text{if } a \in V_M, \\ 1 & \text{if } a = f. \end{cases}$$

This completes the proof. □

3.2 The case where $r = 2$

In this section, we deal with the case where M (see (4)) has two irreducible components of which the two spectral radii are distinct. Lemma 4 below shows that the nonzero 'communication class' C of M ends up being positive after iterations of M if the matrices in the diagonal parts are primitive. Let E_n be $n \times n$ square full matrix, and M be of the form (12), we denote by

$$M^n = \begin{bmatrix} A_n & 0 \\ C_n & B_n \end{bmatrix}$$

the corresponding components of product of M^n.

Lemma 4. *Let*

$$M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \tag{12}$$

be the irreducible decomposition of M, where $A \in \mathbb{R}^{k\times k}$, $B \in \mathbb{R}^{l\times l}$ and $C \in \mathbb{R}^{l\times k}$. If A, B are primitive and $C \neq 0$, then there exists an $N > 0$ such that $C_N > 0$.

Proof. Since $C \neq 0$, there exists $1 \leq i \leq l$ and $1 \leq j \leq k$ such that $C(i, j) = 1$. Since A, B are primitive, let p and $q > 0$ be such that $A^p \geq E_k$ and $B^q \geq E_l$. Note that if $C_{(n)} \neq 0$, then $(CE_k)_{(n)} > 0$ and, similarly, if $C^{(m)} \neq 0$, then $(E_lC)^{(m)} > 0$. Therefore, if $\kappa - 1 \geq \max\{p, q\}$,

$$C_\kappa \geq CA_{\kappa-1} + B_{\kappa-1}C = CA^{\kappa-1} + B^{\kappa-1}C \geq CE_k + E_lC.$$

This shows that $(C_k)_{(i)} > 0$ and $(C_k)^{(j)} > 0$. Repeating the same process, we have

$$C_{2\kappa} \geq C_\kappa E_k + E_l C_\kappa \geq E_{l\times k}.$$

Thus, $N = 2\kappa$ is the desired positive integer. This completes the proof. □

Lemma 5 indicates that every component of the left eigenvector of the maximal eigenvalue ρ_M is either 0 or positive.

Lemma 5. *Let the assumptions of Lemma 4 be satisfied. Suppose* $w^t = [w_1, w_2]^t \in \mathbb{R}^{k+l}$ *is the left eigenvector corresponding to* ρ_M*, where* $w_1^t \in \mathbb{R}^k$ *and* $w_2^t \in \mathbb{R}^l$*. Then* $w_i = 0$ *or* $w_i > 0$ *for* $i = 1, 2$.

Proof. Let $w^t = [w_1, w_2]^t \in \mathbb{R}^{k+l}$ be the left maximal eigenvector corresponding to ρ_M, i.e., $w \neq 0$ and $wM = \rho w$. For $m \in \mathbb{N}$, we have

$$\begin{aligned} [\rho_M^m w_1, \rho_M^m w_2] &= wM^m = [w_1, w_2] \begin{bmatrix} A_m & 0 \\ C_m & B_m \end{bmatrix} \\ &= [w_1, w_2] \begin{bmatrix} A^m & 0 \\ C_m & B^m \end{bmatrix} \\ &= [w_1 A^m + w_2 C_m, w_2 B^m]\,. \end{aligned} \tag{13}$$

Suppose $\rho_M = \rho_B$, since B is primitive, the Perron-Frobenius theory (cf. [8, 12]) is applied to show that $w_2 > 0$. Take $m = 2(\max\{p, q\} + 1)$, where p and q are defined in Lemma 4, then Lemma 4 is applied to show that $C_m > 0$. Since $w_2 > 0$, we have $\rho_M^m w_1 = w_1 A^m + w_2 C_m \geq w_2 C_m > 0$, thus $w_1 > 0$. Suppose $\rho_M = \rho_A$, it follows from (13) we obtain

$$[w_1, w_2] = \left[\frac{w_1 A^m + w_2 C_m}{\rho_M^m}, \frac{w_2 B^m}{\rho_M^m} \right] = \left[\frac{w_1 A^m + w_2 C_m}{\rho_A^m}, \frac{w_2 B^m}{\rho_A^m} \right]. \tag{14}$$

Since $\rho_B < \rho_A$, we have

$$\frac{w_2 B^m}{\rho_M^m} = \frac{w_2 B^m}{\rho_A^m} \to 0 \text{ as } m \to \infty,$$

Thus, $w_2 = 0$ and it follows from (14) we can deduce that w_1 is the Perron vector of A. Since A is primitive, Perron-Frobenius Theorem is applied to show that $w_1 > 0$. This completes the proof. □

Lemma 5 may be indicated more specifically in Lemma 6. Since the proof is almost identical to the one from Lemma 5, it is omitted.

Lemma 6. *Under the assumptions of Lemma 4, suppose* $w^t = [w_1, w_2]^t \in \mathbb{R}^{k+l}$ *is the left maximal eigenvector corresponding to* ρ_M*. Then*

1. *If* $1 < \rho_A < \rho_B$*, then* $w = [w_1, w_2] > 0$*, and* w_2 *is the left maximal eigenvector of* ρ_B*.*

2. *If* $\rho_A > \rho_B > 1$*, then* $w = [w_1, 0]$*, and* $w_1 > 0$ *is the positive left maximal eigenvector corresponding to* ρ_A*.*

The same proof can be applied to the right maximal eigenvector of $\mathbf{M}$ as well.

Lemma 7. *Under the same assumption of Lemma 6, suppose $v = [v_1^t, v_2^t]^t \in \mathbb{R}^{k+l}$ is the right maximal eigenvector of M, where $v_1 \in \mathbb{R}^k$ and $v_2 \in \mathbb{R}^l$. Then*

1. *If $1 < \rho_A < \rho_B$, then $v_1 = 0$ and $v_2 > 0$ is the right maximal eigenvector of ρ_B.*
2. *If $\rho_A > \rho_B > 1$, then $0 < v = [v_1^t, v_2^t]^t \in \mathbb{R}^{k+l}$ and v_1 is the right maximal eigenvector of ρ_A.*

Proof. The proof is almost identical to Lemma 6 if one replaces left maximal eigenvector $w = [w_1, w_2]$ by the right maximal eigenvector $v = [v_1^t, v_2^t]^t$, and thus we omit it. $\square$

Theorem 8 ($r = 2$)**.** *Let $S = \{p_i\}_{i=1}^L$ be a 1-spread model and $S^f = \{q_i\}_{i=1}^L$ be the associated spread model with a frozen symbol $f \in \mathcal{A} = \{a_i\}_{i=1}^K$. Let the ξ^f-matrix $\mathbf{M}$ be of the form (4) and*

$$\mathbf{M} = \begin{bmatrix} A & 0 & 0 \\ C & B & 0 \\ D & E & 1 \end{bmatrix} = \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix} \text{ and } M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

be the irreducible decomposition of $M \in \mathbb{R}^{(k+l)\times(k+l)}$. Suppose A and B are primitive and $C \neq 0$ and set

$$M_1 := \begin{bmatrix} B & 0 \\ E & 1 \end{bmatrix}.$$

Then

1. *If $1 < \rho_A < \rho_B$, we have*
$$s_b^f(a) = \begin{cases} 0 & \text{if } b \in V_A,\ a \in V_A, \\ \bar{v}_{\mathbf{M}}(a) & \text{if } b \in V_A,\ a \in V_B \cup \{f\}, \\ 0 & \text{if } b \in V_B,\ a \in V_A, \\ \bar{v}_{M_1}(a) & \text{if } b \in V_B,\ a \in V_B \cup \{f\}. \end{cases} \tag{15}$$
Furthermore, $s_f^f(a) = 0$, $\forall a \in V_A \cup V_B$ and $s_f^f(f) = 1$.

2. *If $\rho_A > \rho_B > 1$, we have*
$$s_b^f(a) = \begin{cases} \bar{v}_{\mathbf{M}}(a) & \text{if } b \in V_A,\ a \in V_A \cup V_B \cup \{f\}, \\ 0 & \text{if } b \in V_B,\ a \in V_A, \\ \bar{v}_{M_1}(a) & \text{if } b \in V_B,\ a \in V_B \cup \{f\}, \end{cases} \tag{16}$$

Furthermore, $s_f^f(a) = 0$, $\forall a \in V_A \cup V_B$ *and* $s_f^f(f) = 1$.

Remark 9. *If the matrix M is no longer primitive but has the irreducible decomposition of the form*

$$M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

where A and B are primitive with maximal eigenvalues greater than 1 *and if the population is initiated with an individual of type b, then Theorem 8 reveals the following:*

1. *According to the form of the ξ^f-matrix $\mathbf{M}$, individuals of type $b \in V_B$ will not produce individuals of type $a \in V_A$. So, if b is in V_B, then the spread rate of type a with a belonging to V_A is zero. So, the group of the individuals of type a will survive only when $a \in V_B \cup \{f\}$ and the spread rate can be found from the corresponding component in the normalized right eigenvector $\bar{v}_{M_1}(a)$ of the matrix M_1 defined in the Theorem. This is a different result from that in the case when M is primitive.*

2. *If $b \in V_A$ and $a \in V_B \cup \{f\}$ then the spread rate of type a is the ath component in the normalized right eigenvector $\bar{v}_{\mathbf{M}}(a)$ of the ξ^f-matrix $\mathbf{M}$.*

3. *On the other hand, if $b \in V_A$ but $a \in V_A$, then we need to compare the maximal eigenvalues ρ_A and ρ_B of the matrices A and B, respectively, to know the spread rate of type a. Theorem 8 says that, only when $\rho_A > \rho_B$, the individuals of type a will survive with a positive spread rate $\bar{v}_{\mathbf{M}}(a)$; otherwise, the group of the individuals of type a will die out eventually.*

Proof. It suffices to prove Theorem 8 (1), and the other part can be treated similarly. We note that $K = k + l + 1$, and suppose $\mathbf{w} = [w_1, c_1] \in \mathbb{R}^{k+l+1}$ is the left maximal eigenvector of $\mathbf{M}$. Since $1 < \rho_A < \rho_B$, we have $\rho_M = \rho_B$ and $w_1 = w_M$. If we write $\mathbf{w} = [w_M, c_1]$, it follows from Lemma 6 that we have $w_M > 0$. Suppose $\mathbf{M} = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$ is the Jordan decomposition of $\mathbf{M}$. Using the same argument as proposition 2, for all $1 \le i, j \le k + l$, we obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\rho^n} &= \lim_{n\to\infty} \frac{e_j^t \mathbf{M}^n e_i}{\rho_B^n} = \lim_{n\to\infty} \frac{e_j^t \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1} e_i}{\rho_B^n} = c_4 v_{\mathbf{M}}(j), \\ \lim_{n\to\infty} \frac{\mathbf{1}_K \mathbf{M}^n e_i}{\rho^n} &= \lim_{n\to\infty} \frac{\mathbf{1}_K \mathbf{M}^n e_i}{\rho_B^n} = c_4 \sum_{i=1}^{K} v_{\mathbf{M}}(i). \qquad (17) \end{aligned}$$

where $0 < c_4 \in \mathbb{R}$ is a constant. Suppose $\mathbf{v} = [v_1, v_2, d]^t \in \mathbb{R}^K$ is the right maximal eigenvector of $\mathbf{M}$, it is easily seen that $[v_1, v_2]^t$ is also the right maximal eigenvector of M and $d \neq 0$. Thus, the cases where $b \in V_A$ in (15) follows from (17), Lemma 6, and Lemma 7. Since $\xi^f(b)$ cannot produce type a for $b \in V_B$, $a \in V_A$, thus $s_b^f(a) = 0$. The formula for $b \in V_B$ and $a \in V_B \cup \{f\}$ is discussed in Proposition 2. Finally, the cases where $s_f^f(a) = 0$, $\forall a \in V_A \cup V_B$ and $s_f^f(f) = 1$ is also discussed in Proposition 2 as well. This completes the proof of Theorem 8 (1), and the proof is thus completed. □

The following theorem illustrates that if some type f is no longer infectious, then the rate of spreading decays exponentially.

Theorem 10. *Let $\mathcal{S} = \{p_i\}_{i=1}^L$ be a spread model and $\mathcal{S}^f = \{q_i\}_{i=1}^L$ be the 1-spread model with a frozen symbol $f \in \mathcal{A} = \{a_i\}_{i=1}^K$. Suppose $a, b \in V_A \cup V_B$ with $p \in \mathcal{S} \cap \mathcal{S}^f$ and $p^{(0)} = b$ and $O_a(\tau_p^f|_{\Sigma^n(\tau_p^f)}) > 0$, then there exists C_1 and $C_2 > 0$*

$$C_2 \left(\frac{\rho}{\rho_0}\right)^n \leq \frac{O_a(\tau_p^f|_{\Sigma^n(\tau_p^f)})}{O_a(\tau_p|_{\Sigma^n(\tau_p)})} \leq C_2 \left(\frac{\rho}{\rho_0}\right)^n. \tag{18}$$

Furthermore,

$$\lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\Sigma^n(\tau_p^f)})}{O_a(\tau_p|_{\Sigma^n(\tau_p)})} = 0 \ \forall a, b \in V_A \cup V_B. \tag{19}$$

Remark 11. *Theorem 10 tells that as long as the individuals of type a survive in the frozen model with the initial type b, where $a, b \in V_A \cup V_B$, its occurrence, comparing with that in the original non-frozen model, will decay exponentially. More precisely, the decay rate of the ratio of the occurrence after freezing type f to the occurrence before freezing type f is related to the ratio of the maximal eigenvalue ρ of the ξ^f-matrix in the frozen model to the maximal eigenvalue ρ_0 of the ξ-matrix in the original model. From this result, we can know in advance how to choose a proper type f to freeze in order to have a greatest decay rate of a certain type a by computing the eigenvalues of the ξ^f-matrix in the frozen models with different types f. In practice, for example, it can serve as a tool to determine which group of people are put in quarantine in order to have the greatest decay of the number of a certain group of people such as the group of confirm cases.*

Proof. Following the same argument as Theorem 8 with the assumption $O_a(\tau_p^f|_{\Sigma^n(\tau_p^f)}) > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\Sigma^n(\tau_p^f)})}{\rho^n} &= \lim_{n\to\infty} \frac{e_a^t \mathbf{M}^n e_b}{\rho^n} = \mathbf{v}(a) > 0, \\
\lim_{n\to\infty} \frac{O_a(\tau_p|_{\Sigma^n(\tau_p)})}{\rho_0^n} &= \lim_{n\to\infty} \frac{e_a^t \mathbf{M}^n e_b}{\rho_0^n} = \mathbf{v}_0(a) > 0,
\end{aligned}$$

where $\mathbf{v}_0$ is the right maximal eigenvector of $\mathbf{M}_0$ and e_c is defined as (6) with the only $1'$s appearing at the cth coordinate, thus (18) follows. Note that $\rho = \rho_M$ where $\mathbf{M} = \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix}$. Since M is also the principle matrix $\mathbf{M}_0$, we have $\rho_0 > \rho_M = \rho$ (Theorem 4.4.7 [8]). Thus, (19) follows from (18) along with the fact that $\frac{\rho}{\rho_0} < 1$. This completes the proof. □

3.3 The case where $r \geq 2$

Let $\mathbf{M}$ and M be of the forms (4) and (24). Conditions (H1) and (H2) on M are defined as follows.

(H1) The matrices $A^{[ii]}$ are primitive and $1 < \rho_{A^{[ii]}} \neq \rho_{A^{[jj]}}$ $\forall 1 \leq i \neq j \leq r$;

(H2) For all $2 \leq i \leq r$, and $1 \leq j < i$, we have $A^{[ij]} \neq 0$.

For $1 \leq i, j \leq r$, we define $M^{[i,j]}$ according to M below

$$M^{[i,j]} = \begin{bmatrix} A^{[ii]} & 0 & 0 & 0 \\ A^{[(i+1)(i)]} & A^{[(i+1)(i+1)]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ A^{[ji]} & \cdots & \cdots & A^{[jj]} \end{bmatrix} \tag{20}$$

Finally, the associated $\mathbf{M}_i$ for $1 \leq i \leq r$ is defined as follows. Set $\mathbf{M}_1 = \mathbf{M}$. For $2 \leq i \leq r$, $\mathbf{M}_i$ is defined in the matrix form of (21).

$$\begin{aligned}
\mathbf{M} &= \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix} \\
&= \begin{bmatrix} A^{[11]} & 0 \\ \mathbf{C}^{[21]} & \mathbf{M}_1 \end{bmatrix} \\
&= \begin{bmatrix} A^{[11]} & 0 & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 & 0 \\ \vdots & \cdots & \ddots & 0 & 0 \\ A^{[j1]} & \cdots & \cdots & A^{[jj]} & 0 \\ \mathbf{C}^{[(j+1)1]} & \cdots & \cdots & \mathbf{C}^{[(j+1)j]} & \mathbf{M}_j \end{bmatrix}.
\end{aligned} \tag{21}$$

Lemma 12. *Let M be defined as in (24) and satisfy (H1) and (H2). If $w = [w_1, \ldots, w_r]$ is the left maximal eigenvector of M, then $w_i = 0$ or $w_i > 0$ $\forall 1 \le i \le r$.*

Proof. The proof is similar to that of Lemma 5, and thus we omit it. □

Lemma 13. *Suppose M satisfies (H1) and (H2) and that $\rho_M = \rho_{A^{[ll]}}$ for some $1 \le l \le r$. Let $w_M = [w_1, w_2, \ldots, w_r]$ and $v_M = [v_1, v_2, \ldots, v_r]^t$ be the maximal left and right eigenvectors of M, respectively, and let $w_M = [w_{[1,i]}, w_{[i+1,r]}]$ (resp. $v_M = [v_{[1,i]}, v_{[i+1,r]}]^t$), where $w_{[m,n]} = [w_m, \ldots, w_n]$ (resp. $v_{[m,n]} = [v_m, \ldots, v_n]^t$). Then*

1. *$w_{[1,l]} > 0$, $w_{[l+1,r]} = 0$ and $w_{[1,l]}$ is the left maximal eigenvector of $M^{[1,l]}$;*
2. *$v_{[1,l-1]} = 0$, $v_{[l,r]} > 0$, and $v_{[l,r]}$ is the right maximal eigenvector of $M^{[l,r]}$.*

Proof. We only prove Lemma 13 (1), since the proof of Lemma 13 (2) is similar. We claim that if $1 < \rho_M = \rho_{A^{[ll]}}$, then $w_i > 0$ for $1 \le i \le l$, and $w_i = 0$ for $l+1 \le i \le r$. For $r = 2$, since $M^{[1,2]} = \begin{bmatrix} A^{[11]} & 0 \\ A^{[21]} & A^{[22]} \end{bmatrix}$ and $A^{[21]} \ne 0$ (cf. (H2)), the result is true according to Lemma 6 and Lemma 7. For $r \ge 2$, let $w_M = [w_{[1,l]}, w_{[l+1,r]}]$, where $w_{[m,n]} = [w_m, \ldots, w_n]$. Then, we have

$$\begin{aligned} &[\rho_{A^{[ll]}} w_{[1,l]}, \rho_{A^{[ll]}} w_{[l+1,r]}] \\ &= w_M \begin{bmatrix} M^{[1,l]} & 0 \\ \mathbf{C}^{[l+1,l]} & M^{[l+1,r]} \end{bmatrix} \\ &= [w_{[1,l]} M^{[1,l]} + w_{[l+1,r]} \mathbf{C}^{[l+1,l]}, w_{[l+1,r]} M^{[l+1,r]}]. \end{aligned}$$

Suppose $w_{[l+1,r]} \ne 0$. Then, there exists w_k for $l+1 \le k \le r$ such that $w_k > 0$ (cf. Lemma 12). It follows from (22) that

$$[w_{[1,l]}, w_{[l+1,r]}] = [\frac{w_{[1,l]} M_m^{[1,l]} + w_{[l+1,r]} \mathbf{C}_m^{[l+1,l]}}{\rho^m_{A^{[ll]}}}, \frac{w_{[l+1,r]} M_m^{[l+1,r]}}{\rho^m_{A^{[ll]}}}], \text{ for } m \in \mathbb{N}.$$

Since $\rho_{M^{[1,r]}} < \rho_{A^{[ll]}}$, we have

$$\lim_{m \to \infty} \frac{w_{[l+1,r]} M_m^{[l+1,r]}}{\rho^m_{A^{[ll]}}} = 0,$$

a contradiction. Thus, $w_{[l+1,r]} = 0$. It follows from (22) and from $w_{[l+1,r]} = 0$ that we have $w_{[1,l]} M^{[1,l]} = \rho_{A^{[ll]}} w_{[1,l]}$, i.e., $w_{[1,l]}$ is the maximal eigenvector of

$M^{[1,l]}$, and

$$\begin{aligned}[\rho_{A^{[ll]}} w_{[1,l-1]}, \rho_{A^{[ll]}} w_l] &= [w_{[1,l-1]}, w_l] \begin{bmatrix} M^{[1,l-1]} & 0 \\ \mathbf{C}^{[l,l-1]} & A^{[l,l]} \end{bmatrix} \\ &= [w_{[1,l-1]} M^{[1,l-1]} + w_l \mathbf{C}^{[l,l-1]}, w_l A^{[l,l]}]. \end{aligned} \tag{22}$$

Clearly, w_l is the maximal eigenvector of $A^{[ll]}$ and since $A^{[ll]}$ is primitive, then $w_l > 0$. The rest of the proof runs the same as Lemma 5, we only sketch the proof. It follows from (22), we obtain

$$w_l \mathbf{C}^{[l,l-1]} = [w_l A^{[l,1]}, w_l A^{[l,2]}, \ldots, w_l A^{[l,l-1]}].$$

Since M satisfies (H2), $A^{[l,j]} \neq 0\ \forall 1 \leq j < l$. Using the same argument as that of Lemma 4 along with the fact that all diagonal blocks of $M^{[1,l]}$ are primitive (cf. (H1)), we deduce that there exists a large $m_l \in \mathbb{N}$ such that $A^{[l,j]}_{m_l}$ is a full matrix $\forall 1 \leq j < l$. This means that

$$\begin{aligned}\rho_{A^{[ll]}_{m_l}} w_{[1,l-1]} &= w_{[1,l-1]} M^{[1,l-1]}_{m_l} + w_l \mathbf{C}^{[l,l-1]}_{m_l} \geq w_l \mathbf{C}^{[l,l-1]}_{m_l} \\ &= [w_l A^{[l,1]}_{m_l}, w_l A^{[l,2]}_{m_l}, \ldots, w_l A^{[l,l-1]}_{m_l}] > 0\end{aligned}$$

This shows that $w_{[1,l-1]} > 0$. This completes the proof. □

Theorem 14 ($r > 2$)**.** *Let $\mathcal{S} = \{p_i\}_{i=1}^L$ be a 1-spread model and $\mathcal{S}^f = \{q_i\}_{i=1}^L$ be the 1-spread model with a frozen symbol $f \in \mathcal{A} = \{a_i\}_{i=1}^K$. Let $\mathbf{M}$ be the associated ξ-matrix of $\mathcal{S}^f$ and of the form (4), where M is as (24). Suppose M satisfies conditions (H1), (H2) and $\rho_M = \rho_{A^{[ll]}}$ for some $1 \leq l \leq r$. Then*

1. *The values $s_b^f(a)$ can be calculated as follows.*

$$s_b^f(a) = \begin{cases} 0 & \text{if } b \in V_{M^{[1,l]}},\ a \in V_{M^{[1,l-1]}}, \\ \bar{v}_{\mathbf{M}_l}(a) > 0 & \text{if } b \in V_{M^{[1,l]}},\ a \in V_{M^{[l,r]}}, \\ 0 & \text{if } b \in V_{M^{[(l+1),r]}},\ a \in V_{M^{[1,l]}}, \end{cases} \tag{23}$$

where $\bar{v}_{\mathbf{M}_l}$ is the positive normalized maximal eigenvector of $\mathbf{M}_l$.

2. *By using the same scheme on $\mathbf{M}_{l+1}$ as above, the values of $s_b^f(a)$ for $b \in V_{M^{[(l+1),r]}}$, $a \in V_{M^{[(l+1),r]}}$ can be computed in the same fashion. Furthermore, $s_f^f(a) = 0\ \forall a \in V_M$ and $s_f^f(f) = 1$.*

Remark 15. *When all the component matrices in the form*

$$M = \begin{bmatrix} A^{[11]} & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ A^{[r1]} & \cdots & \cdots & A^{[rr]} \end{bmatrix} \tag{24}$$

of the ξ-matrix M are primitive (H1) and satisfy (H2) and there is a component $A^{[l,l]}$ which has the maximal eigenvalue $\rho_{A^{[l,l]}}$ the same as the maximal eigenvalue ρ_M of the ξ-matrix M, Theorem 14 says that, if the initial type b is not a frozen type, then the spread rate of type a is positive only when $b \in V_{M^{[1,l]}}$ and $a \in V_{M^{[l,r]}}$. It means that individuals of type a only survive eventually only in the cases where $b \in V_{M^{[1,l]}}$ and $a \in V_{M^{[l,r]}}$.

Proof. The proof is obtained by combining the same proof as that of Theorem 8 and Lemma 13. □

Example 16. *Suppose*

$$\begin{aligned} \mathbf{M} &= \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix} = \begin{bmatrix} A^{[11]} & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 \\ A^{[31]} & A^{[32]} & A^{[33]} & 0 \\ c^{[31]} & c^{[32]} & c^{[33]} & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2+3+2+1}, \end{aligned}$$

where

$$A^{[11]} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix},\ A^{[22]} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A^{[33]} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Clearly, M satisfies the conditions (H1) and (H2). Since $l = 2$, i.e., $\rho_M = \rho_{A^{[22]}} = 3$, Theorem 14 is applied to show that

$$s_b^f(a) = \begin{cases} 0 & \text{if } b \in \{a_1, \ldots, a_5\},\ a \in \{a_1, a_2\}, \\ \bar{v}_{\mathbf{M}_2}(a) > 0 & \text{if } b \in \{a_1, \ldots, a_5\},\ a \in \{a_3, \ldots, a_8\}, \\ 0 & \text{if } b \in \{a_6, a_7\},\ a \in \{a_1, \ldots, a_5\}, \end{cases}$$

where

$$\mathbf{M}_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{3+2+1}.$$

Next, Proposition 2 is applied to calculate the values of $s_b^f(a)$ *for* $b \in \{a_6, a_7\}$ *and* $a \in \{a_6, a_7\}$*. That is*

$$s_b^f(a) = \bar{v}_{\mathbf{M}_3}(a) \textit{ for } b \in \{a_6, a_7\} \textit{ and } a \in \{a_6, a_7\}.$$

where

$$\mathbf{M}_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \in \mathbb{R}^{2+1}.$$

Finally, $s_{a_8}^{a_8}(a) = 0 \ \forall a \in \{a_1, \ldots, a_7\}$ *and* $s_{a_8}^{a_8}(a_8) = 1$.

3.4 1-spread model with a frozen set: single layer

Let $S = \{p_i\}_{i=1}^L$ be a 1-spread model and $F = \{f_i\}_{i=1}^R \subseteq \mathcal{A}$ be a subset of the symbol set $\mathcal{A}$. For $f_i \in F$ and $p_{f_i} \in \mathcal{S}$ with $p_{f_i}^{(0)} = f_i \in F$, we replace the pattern p_{f_i} with $\widehat{p}_{f_i} = (p_{f_i}^{(0)}; p_{f_i}^{(1)}) = (f_i; f_i)$. We write $\mathcal{S}^F = \cup_{i=1}^R ((\mathcal{S} \backslash p_{f_i}) \cup \widehat{p}_{f_i})$ and call it the 1-*spread model with a frozen set* F. Let ξ^F be the associated substitution and $\mathbf{M}_F$ be the corresponding ξ^F-matrix. Similarly, we have

$$\mathbf{M}_F = \begin{bmatrix} M & 0 \\ \mathbf{C} & M_F \end{bmatrix}.$$

Since M_F may not irreducible, thus we assume that

$$M = \begin{bmatrix} A^{[11]} & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ A^{[r1]} & \cdots & \cdots & A^{[rr]} \end{bmatrix} \text{ and } M_F = \begin{bmatrix} B^{[11]} & 0 & 0 & 0 \\ B^{[21]} & B^{[22]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ B^{[s1]} & \cdots & \cdots & B^{[ss]} \end{bmatrix},$$

where $A^{[ii]}$ and $B^{[jj]}$ are irreducible $\forall 1 \leq i \leq r$ and $1 \leq j \leq s$. Suppose $\mathbf{C} \neq 0$, M and M_F satisfy conditions (H1) and (H2), then the values $s_b^F(a)$ for $a, b \in \left(\cup_{i=1}^r A^{[ii]}\right) \cup \left(\cup_{j=1}^s B^{[jj]}\right)$ can be calculate by Theorem 14.

4 1-spread model with a frozen symbol: constant and increasing range

Though the following Lemma appears in [3], we address the proof for readers' convenience.

Lemma 17 (Lemma 2 [3]). *Let $\{a_n\}, \{b_n\}$ be real sequences and $\{b_n\}, \{d_n\}$ be positive real sequences. Suppose*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{c_n}{d_n} = L.$$

Then,

$$\lim \frac{a_n + c_n}{b_n + d_n} = L.$$

Furthermore, suppose that $\lim_{n\to\infty} \sum_{j=1}^n b_j = +\infty$. *Then,*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^n a_j}{\sum_{j=1}^n b_j} = L.$$

Proof. The equality $\lim_{n\to\infty} \frac{a_n+c_n}{b_n+d_n} = L$ is obvious, and thus we only prove the second part. We claim that for every $\bar{m} > L$ and every $m < L$k, we have

$$\begin{aligned} \limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &\leq \bar{m}, \text{ and} \\ \liminf_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &\geq m. \end{aligned}$$

Indeed, since there exists $N_1 \in \mathbb{N}$ such that $\frac{a_n}{b_n} < \bar{m}$ for all $n \geq N_1$, for all $n \geq N_1$ we have $a_n < \bar{m} b_n$ and

$$\begin{aligned} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &= \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + \frac{a_{N_1+1} + \cdots + a_n}{b_1 + \cdots + b_n} \\ &< \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + \bar{m} \frac{b_{N_1+1} + \cdots + b_n}{b_1 + \cdots + b_n} \\ &< \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + \bar{m}. \end{aligned}$$

We note that N_1 is fixed and $\lim_{n\to\infty} \sum_{j=1}^n b_j = +\infty$. Therefore,

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \limsup_{n\to\infty} \left(\frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + \bar{m} \right) = \bar{m}.$$

For the other part, since there exists $N_2 \in \mathbb{N}$ such that $\frac{a_n}{b_n} > m$ for all $n \geq N_2$, using the same argument again, we have

$$\begin{aligned} \liminf_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &\geq \liminf_{n\to\infty} \left(\frac{a_1 + \cdots + a_{N_2}}{b_1 + \cdots + b_n} + m \frac{b_{N_2+1} + \cdots + b_n}{b_1 + \cdots + b_n} \right) \\ &= m. \end{aligned}$$

The proof is thus completed. □

Theorem 18. *Let $\mathcal{S} = \{p_i\}_{i=1}^L$ be a spread model and $\mathcal{S}^f = \{q_i\}_{i=1}^L$ be the spread model with a frozen symbol $f \in \mathcal{A} = \{a_i\}_{i=1}^K$. Suppose $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$ is a sequence of natural numbers with $s_n := \sum_{i=1}^n k_i \to \infty$ as $n \to \infty$. Then, for all $a, b \in \mathcal{A}$, we have*

$$s_b^f(a, \{k_n\}_{n=1}^\infty) = s_b^f(a). \tag{25}$$

In particular, if $k_n = k$ for all $n \in \mathbb{N}$, (25) holds true as well.

Proof. It suffices to prove (25) for the case that $k_n = k$ for all $n \in \mathbb{N}$, since all the rest can be treated similarly. For $p \in \mathcal{S}$ with $p^{(0)} = b \in \mathcal{A}$ and for $a \in \mathcal{A}$, the value $s_b^f(a)$ is calculated. By denoting $\sigma_b^{(i)} = \left|\Sigma^i(\tau_p)\right|$, we have

$$\left|\Delta_{kn}^{k(n+1)}(\tau_p^f)\right| = \sigma_b^{(kn+1)} + \cdots + \sigma_b^{(k(n+1))}.$$

Thus,

$$\begin{aligned} s_b^f(a; \{k_n\}_{n=1}^\infty) &= \lim_{n\to\infty} s_b^f(a; [kn+1, k(n+1)]) \\ &= \lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\Delta_{kn}^{k(n+1)}(\tau_p^f)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_p^f)\right|} \\ &= \lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\Sigma_b^{(kn+1)}}) + \cdots + O_a(\tau_p^f|_{\sigma_b^{(k(n+1))}})}{\sigma_b^{(kn+1)} + \cdots + \sigma_b^{(k(n+1))}} \end{aligned}$$

Let $a_n = O_a(\tau_p^f|_{\sigma_b^{(n)}})$ and $b_n = \sigma_b^{(n)}$ $\forall n \in \mathbb{N}$, then it follows from Lemma 17 and from (26) that we have

$$\begin{aligned} s_b^f(a; \{k_n\}_{n=1}^\infty) &= \lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\sigma_b^{(kn+1)}}) + \cdots + O_a(\tau_p^f|_{\sigma_b^{(k(n+1))}})}{\sigma_b^{(kn+1)} + \cdots + \sigma_b^{(k(n+1))}} \\ &= \lim_{n\to\infty} \frac{O_a(\tau_p^f|_{\sigma_b^{(n)}})}{\sigma_b^{(n)}} = s_b^f(a). \end{aligned}$$

This completes the proof. □

5 m-spread models

In [3], the authors develop a methodology for calculating the spread rate of m-spread models. The method developed in this article allows us to calculate

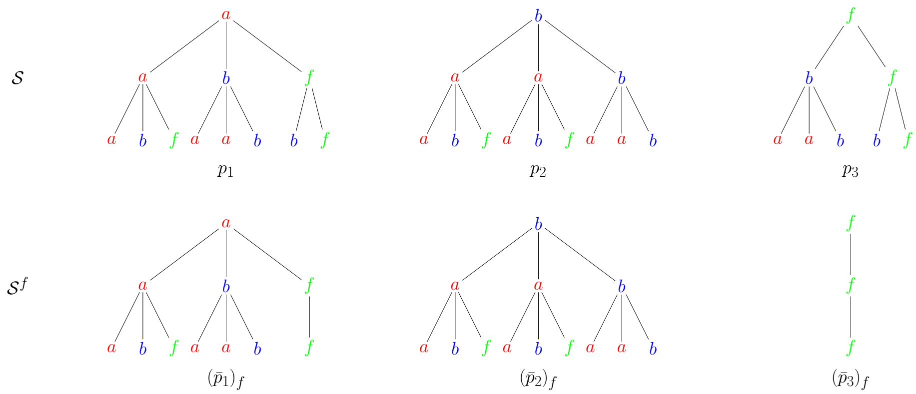


Figure 4: Formation of m-spread model with a frozen symbol f

the spread rate for the m-spread model with frozen symbols. We simply outline the following methodology.

Let $1 \leq m \in \mathbb{N}$, and $\mathcal{A} = \{a_i\}_{i=1}^K$ be a type set. For a finite set $F \subseteq \Delta_m$, a function $p : F \to \mathcal{A}$ is called an *m-spread pattern* (*m-pattern*), and $F = F_p$ is the *support* of p. Let $\mathcal{P}_m$ be the set of m-patterns. We define $p^{(0)} = p|_{\Delta_{m-1}}$ and for all $g \in F_p$ with $|g| = 1$ and $p(g) := p|_{\Delta_{m-1}^g} \in \mathcal{P}_{m-1}$. Following the notations and terminology of 1-spread models, we write $p = (p^{(0)}; p^{(1)})$ as (1), and $\mathcal{S} = \{p_i\}_{i=1}^L \subseteq \mathcal{P}_m$ is called an *m-spread model* if for all $p \in \mathcal{S}$ and $g \in F_p$ with $|g| = 1$, there exists a unique $q \in \mathcal{S}$ such that $q^{(0)} = p(g)$. Suppose $\mathcal{S}$ is an m-spread model and $p \in \mathcal{S}$, the *infinite spread pattern induced from* p, say τ_p, is similarly defined.

Fix $f \in \mathcal{A}$, if f first appears in some $p \in \mathcal{S}$ on $g \in F_p$ and $|g| = r \leq m$. More precisely, $p(g) = f$, and there is no ancestor h of g with $p(h) = f$. The pattern $p|_{\Delta_{m-r}^g}$ is transformed as follows: 1) The set Δ_{m-r}^g is replaced by a path $g = g_r$, $g_{r+1}, \ldots, g_m$ of length $m - r$; and 2) Define all types in this path by f's. We denote the resulting pattern by $\bar{p}_f$. The underlying reason for defining $\bar{p}_f$ is that if $p(g) = f$ and f becomes non-infectious, then the type f cannot produce any other type than f. It is easily seen that the modified $\mathcal{S}^f := \{(\bar{p}_i)_f\}_{i=1}^L$ is an m-spread model, and we call it the *m-spread model with a frozen symbol f*. This process is illustrated in Figure 4.

To calculate the spread rates (2) for an m-spread model, a method that transforms an m-spread model to a 1-spread model (call it *induced model*) is proposed. For an m-spread model $\mathcal{S}$, we define a new type set $\mathbb{A}$ by collecting $\alpha \in \mathcal{P}_{m-1}$ in which α appears in $F_p|_{\Delta_{m-1}}$ or $F_p|_{\Delta^g_{m-1}}$ for some $p \in \mathcal{P}_m \cap \mathcal{S}$ and $g \in F_p$ with $|g| = 1$. For $q \in \mathcal{P}_{m-1}$, we denote by $\alpha(q) \in \mathbb{A}$ the corresponding transformation from $\mathcal{P}_{m-1}$ to $\mathbb{A}$. For the type set $\mathbb{A}$, and $p = (p^{(0)}; p(g_1), \ldots, p(g_{d_p^{(1)}})) \in \mathcal{S}$, the associated '1-spread model' is defined as

$$\mathbb{S} := \{\widehat{p}_i = (\alpha(p_i^{(0)}); \alpha(p_i(g_1)), \ldots, \alpha(p_i(g_{d_{p_i}^{(1)}})))\}_{i=1}^{L}.$$

Clearly, $\mathbb{S} = \{\widehat{p}_i\}_{i=1}^{L}$ is an 1-spread model with the type set $\mathbb{A}$. Suppose $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ and $s_n = \sum_{n=1}^{n} k_i \to \infty$ as $n \to \infty$. The formula for the spread rate of type a in $\tau_{\widehat{p}}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\widehat{p}})}$ is established (Theorem 4 of [3]):

$$s_{\widehat{p}}(a, \{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} \frac{O_a(\tau_{\widehat{p}}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\widehat{p}})})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_{\widehat{p}})\right|} = \sum_{\alpha\in\theta(a)} v(\alpha), \tag{26}$$

where v is the right maximal eigenvector of the primitive ξ-matrix $\mathbf{M}$, and

$$\theta(a) = \{\alpha \in \mathbb{A} : \ \alpha = \alpha(p^{(0)}) \in \mathcal{P}_{m-1} \text{ and } p^{(0)}(\epsilon) = a\}.$$

If $\mathcal{S}^f := \{(\bar{p}_i)_f\}_{i=1}^{L}$ is the m-spread model with a frozen symbol f, two conditions proposed in Section 2.2 are no longer true, so the associated ξ-matrix $\mathbf{M} = \mathbf{M}^f$ is not a primitive matrix. However, combining Theorem 18, Theorem 14 and the discussion in Section 3.4[2], our method still works after rearranging $\mathbf{M}$ in the form of the irreducible components and combining it with formula (26).

6 Numerical results

This section is devoted to providing examples of spread models with different structures in terms of irreducible decomposition. More specifically, each of these examples represents one of the following three classes of spread models:

1. all $A^{[ii]}$'s are primitive matrices and all $\rho_{A^{[ii]}}$'s are distinct;
2. all $A^{[ii]}$'s are primitive matrices and some $\rho_{A^{[ii]}}$'s are coincident;

[2]In general, the ξ-matrix $\mathbf{M}$ is not necessarily of the form (4), however, it must be of the form of a certain ξ-matrix derived from a 1-spread model with a frozen set.

3. some $A^{[ii]}$'s are irreducible but not primitive.

In particular, these examples cover not only the class discussed in the previous sections (item 1) but the classes under more general settings (items 2 and 3).

6.1 Experiment 1: primitive components, distinct spectral radii

Let $\mathcal{A} = \{1,2,3,4,5,6,7\}$ be a type set of a spread model $\mathcal{S}$, $\mathbf{M}$ be the associated ξ-matrix of $\mathcal{S}$ that is defined as

$$\mathbf{M} = \begin{bmatrix} A & O & O \\ C & B & O \\ D & E & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right],$$

It is to be noted that this matrix satisfies (H1) and (H2). By setting

$$M_1 := \begin{bmatrix} B & 0 \\ E & 1 \end{bmatrix},$$

we have, as a consequence of Theorem 8, that

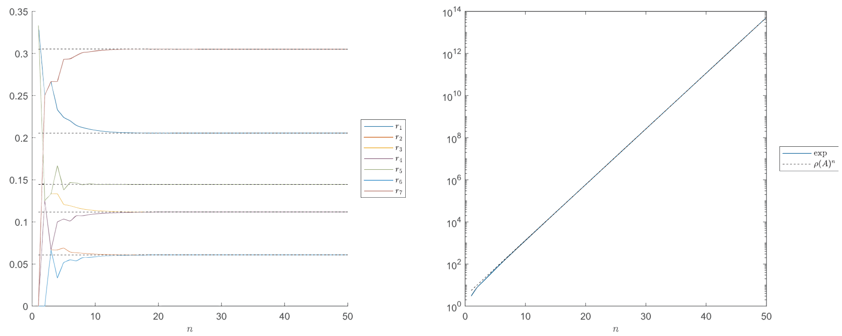
$$s_b(a) = \begin{cases} \bar{v}_{\mathbf{M}}(a) & \text{if } b \in \{1,2,3\}, a \in \mathcal{A}, \\ 0 & \text{if } b \in \{4,5,6\}, a \in \{1,2,3\}, \\ \bar{v}_{M_1}(a) & \text{if } b \in \{4,5,6\}, a \in \{4,5,6,7\} \cup \{f\}, \\ 0 & \text{if } b \in \{7\}, a \in \{1,2,3,4,5,6\}, \\ 1 & \text{if } b \in \{7\}, a \in \{7\}. \end{cases} \tag{27}$$

Note that the right eigenvectors $v_{\mathbf{M}}$ and v_{M_1}, associated with the eigenvalues $\rho(\mathbf{M})$ and $\rho(M_1)$, respectively, can be chosen such that

$$v_{\mathbf{M}} = \begin{bmatrix} I \\ (\rho_A I - B)^{-1} C \\ (\rho_A - 1)^{-1}(D + E(\rho_A I - B)^{-1} C) \end{bmatrix} v_A,$$

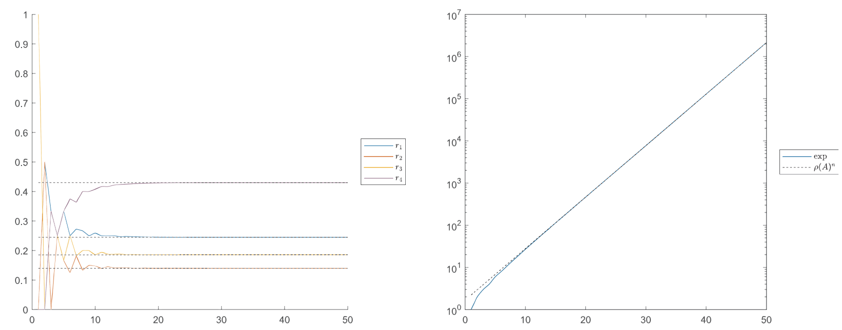
$$v_{M_1} = \begin{bmatrix} I \\ (\rho_B - 1)^{-1} E \end{bmatrix} v_B,$$

which are clearly positive vectors since A, B are primitive matrices, $\rho_A > \rho_B > 1$, and $(I - \frac{B}{\rho_A})^{-1} = \sum_{i=0}^{\infty} (\frac{B}{\rho_A})^i$ is positive. Given the ancestor b, the spread rates $s_b(a)$ as well as the sizes of the population are illustrated in Figure 5, 6, and 7, which corresponds to the cases $b \in \{1,2,3\}$, $b \in \{4,5,6\}$, and $b \in \{7\}$



(a) spread rate (b) population size

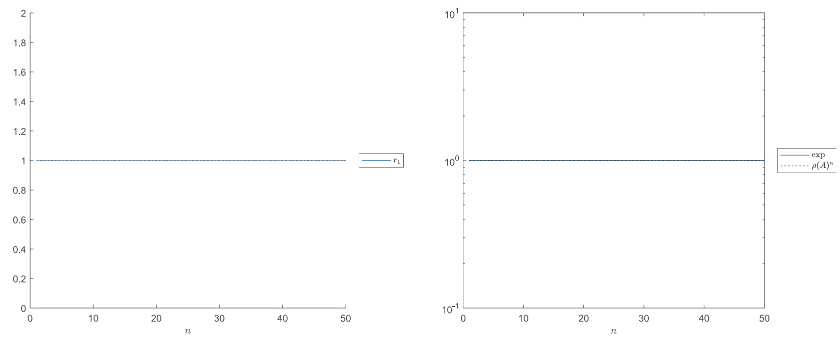
Figure 5: Experiment 1: spread rate $s_b(a)$ for $b \in \{1, 2, 3\}$



(a) spread rate (b) population size

Figure 6: Experiment 1: spread rate $s_b(a)$ for $b \in \{4, 5, 6\}$

accordingly. As is consistent with Theorem 8, the convergence of the spread rates to the normalized eigenvectors $\bar{v}_{\mathbf{M}}$, $\bar{v}_{M_1}$, and 1 is observed in the figures, and the slope of the log population is seen to be $\log \rho_A$, $\log \rho_B$, and 0.



(a) spread rate (b) population size

Figure 7: Experiment 1: spread rate $s_b(a)$ for $b \in \{7\}$

6.2 Experiment 2: primitive components, coincident spectral radii

Let $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7\}$ be a type set of a spread model $\mathcal{S}$ and $\mathbf{M}$ be the associated ξ-matrix of $\mathcal{S}$ that is defined as

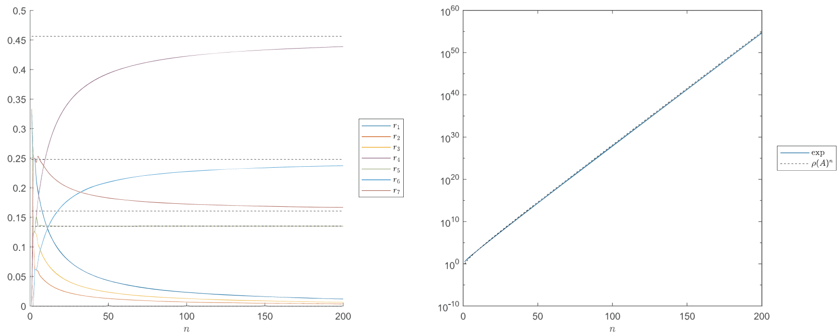
$$\mathbf{M} = \begin{bmatrix} A & O & O \\ C & B & O \\ D & E & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{array}\right].$$

Though (H2) is still satisfied in this case, (H1) is clearly not true, since $A = B$. Let

$$M_1 := \begin{bmatrix} B & 0 \\ E & 1 \end{bmatrix},$$

v_A and w_A (respectively, v_B and w_B) be the right and left eigenvectors of A (respectively, B) associated with $\rho_{\mathbf{M}}$ such that $w_A^T v_A = 1$ (respectively, $w_B^T v_B = 1$). Note that $\mathbf{M}$ and M_1 each has exactly one left and one right eigenvector (up to rescaling) associated with $\rho_{\mathbf{M}}$. Let the right eigenvectors $v_{\mathbf{M}}$ and v_{M_1} associated with $\rho_{\mathbf{M}} = \rho_{M_1}$ be chosen as

$$v_{\mathbf{M}} = \begin{bmatrix} O \\ v_B w_B^T C \\ (\rho_A - 1)^{-1} E v_B w_B^T C \end{bmatrix} v_A \text{ and } v_{M_1} = \begin{bmatrix} I \\ (\rho_B - 1)^{-1} E \end{bmatrix} v_B.$$



(a) spread rate (b) population size

Figure 8: Experiment 2: spread rate $s_b(a)$ for $b \in \{1, 2, 3\}$

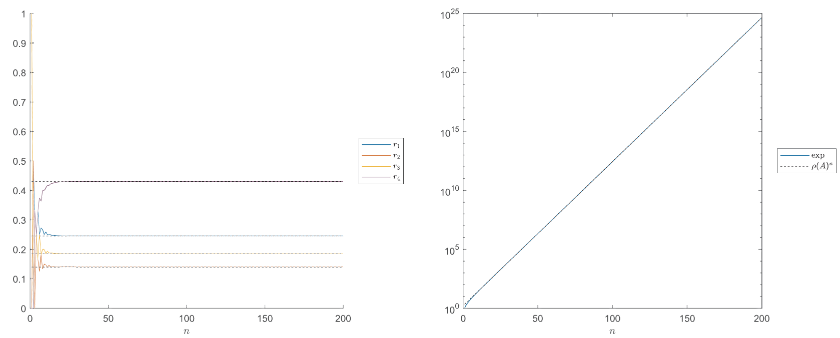
Then, since $\mathbf{M}$ has a unique Jordan block associated with $\rho_{\mathbf{M}}$, whose size is 2, one can show that $\frac{\mathbf{M}^n}{n\rho_{\mathbf{M}}^{n-1}}$ converges as n tends to infinity. Furthermore, by a straightforward block-wise estimation of $\frac{\mathbf{M}^n}{n\rho_{\mathbf{M}}^{n-1}}$, one derives that

$$\lim_{n\to\infty} \frac{1}{n\rho_{\mathbf{M}}^{n-1}} \cdot \mathbf{M}^n = v_{\mathbf{M}} \left[\begin{array}{ccc} w_A^T & O & O \end{array} \right].$$

We should note that $v_{\mathbf{M}}(i)$ and $v_{M_1}(i)$ are positive if $i \in \{4, 5, 6, 7\}$ since v_A, v_B, w_A, and w_B are all positive and $(I - \frac{B}{\rho_A})^{-1} = \sum_{i=0}^{\infty} (\frac{B}{\rho_A})^i$ is positive. It then follows that

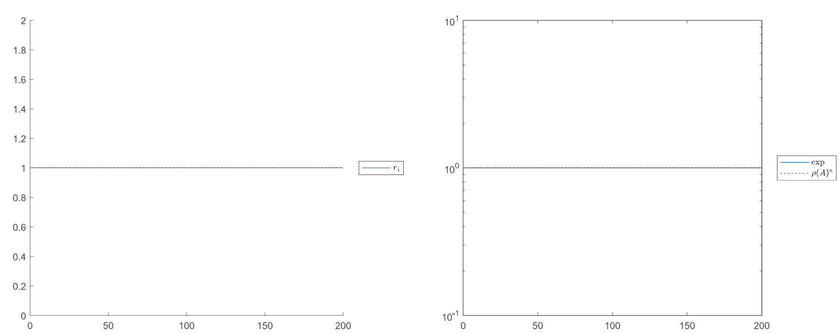
$$s_b(a) = \begin{cases} \bar{v}_{\mathbf{M}}(a) & \text{if } b \in \{1,2,3\}, a \in \mathcal{A}, \\ 0 & \text{if } b \in \{4,5,6\}, a \in \{1,2,3\}, \\ \bar{v}_{M_1}(a) & \text{if } b \in \{4,5,6\}, a \in \{4,5,6,7\} \cup \{f\}, \\ 0 & \text{if } b \in \{7\}, a \in \{1,2,3,4,5,6\}, \\ 1 & \text{if } b \in \{7\}, a \in \{7\}. \end{cases} \tag{28}$$

Given the ancestor b, the spread rates $s_b(a)$ as well as the sizes of the population are illustrated in Figure 8, 9, and 10, which corresponds to the case $b \in \{1, 2, 3\}$, $b \in \{4, 5, 6\}$, and $b \in \{7\}$ accordingly. As is consistent with Theorem 8, the convergence of the spread rates to eigenvectors $\bar{v}_{\mathbf{M}}$, $\bar{v}_{M_1}$, and 1 are observed in the figures, and the slope of the log population is seen to be $\log \rho_{\mathbf{M}}$, $\log \rho_{\mathbf{M}}$, and 0.



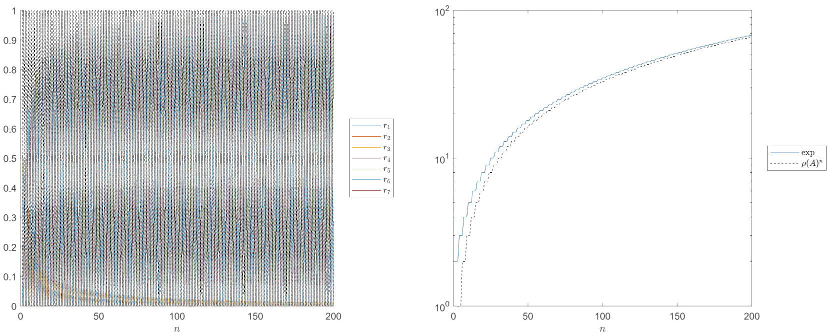
(a) spread rate (b) population size

Figure 9: Experiment 2: spread rate $s_b(a)$ for $b \in \{4, 5, 6\}$



(a) spread rate (b) population size

Figure 10: Experiment 2: spread rate $s_b(a)$ for $b \in \{7\}$



(a) spread rate (b) population size

Figure 11: Experiment 3: spread rate $s_b(a)$ for $b \in \{1, 2, 3\}$

6.3 Experiment 3: irreducible components

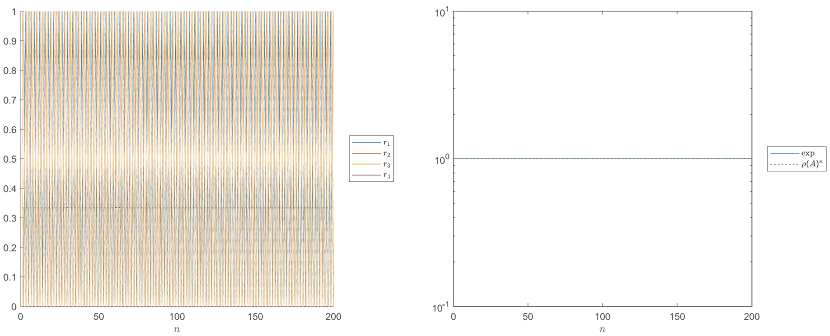
Let $\mathcal{A} = \{1, 2, 3, 4, 5, 6, 7\}$ be a type set of a spread model $\mathcal{S}$ and $\mathbf{M}$ be the associated ξ-matrix of $\mathcal{S}$ that is defined as

$$\mathbf{M} = \begin{bmatrix} A & O & O \\ C & B & O \\ D & E & 1 \end{bmatrix} = \left[\begin{array}{ccc|ccc|c} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right].$$

Since $A = B^T$ is an irreducible matrix of period 3 and $D = O$, the hypotheses (H1) and (H2) are clearly not satisfied. As a result, one observes that $\mathbf{M}^3$ falls within the class of the previous section. This therefore leads to a non-convergent spread rate $s_b(a)$, as is shown in Figure 11, 12, and 13, which corresponds to the cases $b \in \{1, 2, 3\}$, $b \in \{4, 5, 6\}$, and $b \in \{7\}$, respectively. In fact, it is seen in these figures that the period of the spread rate coincides with the period of the matrices, and the sizes of the population have polynomial growth.

7 Conclusion and open questions

As we stated in the introduction, working with a spread model with frozen symbols is essential in the decision making for the disease control when we want to predict what happens to the spread rates of other existing types after



(a) spread rate (b) population size

Figure 12: Experiment 3: spread rate $s_b(a)$ for $b \in \{4, 5, 6\}$

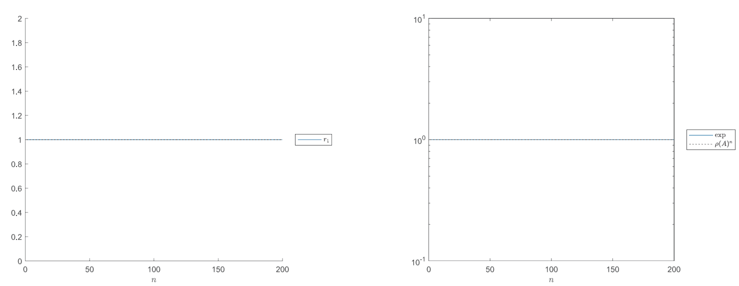


Figure 13: Experiment 3: spread rate $s_b(a)$ for $b \in \{7\}$

blocking certain type. In this article we investigate the spread model with a frozen symbol and frozen sets and several results are proved. First of all, in Proposition 2 (1 irreducible component), Theorem 8 (2 irreducible components) and Theorem 14 (r irreducible components), we give complete characterizations of the spread rate for the 1-spread model with a frozen symbol within a single layer. Secondly, we extend the preceding results to the case where the spread rate is calculated with a constant or increasing ranges (Theorem 18). Finally, we also discuss the m-spread model with frozen symbols in Section 5.

The significance of our work is that we derive a method in Section 3 to deal with the irreducible components in the ξ-matrix. In the classic theory of substitution, the associated ξ-matrices are assumed to be primitive so that some beautiful results can be established. But, often when we consider a spread model with frozen symbols, the associated ξ^f-matrix of the frozen model is no longer primitive although the original ξ-matrix of the original spread model is primitive. Therefore, how to deal with the non-primitive components in the ξ^f-matrix in order to arrive at the solutions we are looking for becomes essential in the whole work.

However, the results we established in this paper are far from being conclusive, we still have some more work needed to be done before we fully comprehend the general cases. Here, we list problems below for the future study.

Problem 19. *Conditions (H1) and (H2) should be removed in general situations. However, there remain following issues which needs to be addressed.*

1. *There exists* $1 \leq i \neq j \leq r$ *with* $1 < \rho_{A^{[ii]}} = \rho_{A^{[jj]}}$.
2. *For all* $1 \leq i \leq r$, $A^{[ii]}$ *are not all primitive, e.g., irreducible.*
3. *There exists* $A^{[ij]} = 0$, *for some* $2 \leq i \leq r$, *and* $1 \leq j < i$.

Moreover, when we start to consider the randomness of the spread model as we did in our previous work [2] and [3], the following problem becomes interesting and it will be discussed in our next paper.

Problem 20. *What is the proper setting for the random spread model with frozen symbols and what happens to the new spread rate? How does it relate to the topological spread model with frozen symbols?*

Data Availability

The data that support the findings of this study are available within the article.

Acknowledgment

We would especially like to thank the anonymous reviewers for their valuable information and comments which help us to improve the quality of the manuscript. This work was supported by the Ministry of Science and Technology, Taiwan [grant numbers MOST 109-2115-M-004-002-MY2, MOST 109-2115-M-004-001-MY2].

References

[1] M. E. Alexander, C. Bowman, S. M. Moghadas, R. Summers, A. B. Gumel, and B. M. Sahai, *A vaccination model for transmission dynamics of influenza*, SIAM Journal on Applied Dynamical Systems **3** (2004), no. 4, 503–524.

[2] J.-C. Ban, C.-H. Chang, J.-I. Hong, and Y.-L. Wu, *Mathematical analysis of spread models: From the viewpoints of deterministic and random cases*, Chaos, Solitons & Fractals **150** (2021), 111106.

[3] ______, *Mathematical analysis of topological and random m-order spread models*, preprint (2022).

[4] L. Billings, E. M. Bollt, and I. B. Schwartz, *Phase-space transport of stochastic chaos in population dynamics of virus spread*, Physical Review Letters **88** (2002), no. 23, 234101.

[5] S. A. Gourley and Y.-J. Lou, *A mathematical model for the spatial spread and biocontrol of the asian longhorned beetle*, SIAM Journal on Applied Mathematics **74** (2014), no. 3, 864–884.

[6] K. S. Halle, C.-W. Wu, M. Itoh, and L. O. Chua, *Spread spectrum communication through modulation of chaos*, International Journal of Bifurcation and Chaos **3** (1993), no. 02, 469–477.

[7] Y. Jin and X.-Q. Zhao, *Spatial dynamics of a periodic population model with dispersal*, Nonlinearity **22** (2009), no. 5, 1167.

[8] D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

[9] J. H. Merkin, D. J. Needham, and B. D. Sleeman, *A mathematical model for the spread of morphogens with density dependent chemosensitivity*, Nonlinearity **18** (2005), no. 6, 2745.

[10] World Health Organization, *WHO COVID-19: Case definitions: updated in Public health surveillance for COVID-19, 22 July 2022*, Technical documents, 2022, `https://apps.who.int/iris/handle/10665/360579`.

[11] C.-H. Ou and J.-H. Wu, *Spatial spread of rabies revisited: influence of age-dependent diffusion on nonlinear dynamics*, SIAM Journal on Applied Mathematics **67** (2006), no. 1, 138–163.

[12] M. Queffélec, *Substitution dynamical systems-spectral analysis*, vol. 1294, Springer, 2010.

[13] S.-G. Ruan and J.-H. Wu, *Modeling spatial spread of communicable diseases involving animal hosts*, Spatial ecology (2009), 293–316.

[14] W.-D. Wang and X.-Q. Zhao, *Basic reproduction numbers for reaction-diffusion epidemic models*, SIAM Journal on Applied Dynamical Systems **11** (2012), no. 4, 1652–1673.

[15] H. Zheng and A. Bonasera, *Chaos, percolation and the coronavirus spread: a two-step model*, The European Physical Journal Plus **135** (2020), no. 10, 1–24.

[16] ______, *Controlling the worldwide chaotic spreading of COVID-19 through vaccinations*, Journal of Modern Physics **13** (2022), no. 01, 1–15.