Topological and random spread models with frozen symbols

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Abstract

When a symbol or a type has been "frozen" (namely, a type of which an individual only produces one individual of the same type), its spread pattern will be changed and this change will affect the long-term behavior of the whole system. However, in a frozen system, the ξ-matrix and the offspring mean matrix are no longer primitive so that the Perron-Frobenius theorem can not be applied directly when predicting the spread rates. In this paper, our goal is to characterize these key matrices and to analyze the spread rate under more general settings both in the topological and random spread models with frozen symbols. More specifically, We propose an algorithm for explicitly computing the spread rate and relate the rate with the eigenvectors of the ξ-matrix or offspring mean matrix. In addition, we reveal that the growth of the population is exponential and that the composition of the population is asymptotically periodic. Furthermore, numerical experiments are provided as supporting evidence for the theory.

When a pandemic happens, modelling the spread of disease is always an important method to predict the future situation and plays a key role in disease control, prevention and decision making. In this manuscript, we propose two mathematical models called spread models with frozen symbols from the topological and random perspectives, respectively. These models can be used to describe the phenomenon after some prevention and control measures are adopted and the viral spread pattern is changed. We have found the long-term behavior of

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the spread rate for each type in these models. In particular, we derive a method to deal with the non-primitive ξ-matrix and offspring mean matrix while the primitive property is usually the key sufficient condition to study the limit behavior of the matrix using its maximal eigenvalue and the corresponding eigenvector in the classic theories.

1 Introduction

1.1 Motivations

Pandemic transmission has been a recurring theme in research in recent years and is a topic that has been garnering much public attention, owing largely to the potential socioeconomic impacts accompanying the spread of infectious diseases. Numerous models have been proposed in the literature to explain and predict the spread of the diseases, which has substantially benefited the governing bodies charged with disease control and public health measures. This is clearly seen in the case of COVID-19 after its outbreak in late 2019.

For a better grasp of both the short-term and long-term behavior of the spread of the coronavirus, techniques from multiple fields of study have been applied, including those in machine learning [1] as well as random stochastic models [7, 10], in which stochastic phenomena are present and sensitive to parameters as well as to initial conditions. Many of the works investigating the spread of the coronavirus incorporate a type of model that classifies individuals into the three major categories: susceptible, infected, and recovered. This so-called SIR model along with its variants has proven effective in investigating the dynamics of the number or fraction of individuals in each category, and various aspects of the model are explored in the hope of arriving at a better conception of disease transmission. In particular, the *basic reproduction number*, that is, the expected number of infected cases generated by a single existing case, is closely related to the stability as well as equilibria of the systems. This number is therefore widely studied under different model assumptions, with data usually compared before and after containment measures have been taken [6, 9, 11, 13, 14, 15, 16, 17, 18] to explain any drop in the number of infected cases. This ease of pandemic is also observed using alternative models. For example, the logistic model considered in [8, 19, 20] aptly portrays the exponential decay of the increase in cases. Similar characteristics are also qualitatively captured by the spread model considered in the authors' previous work [5] by introducing the notion of frozen symbols under the assumption that the offspring matrix is primitive. Nevertheless, the decay of the infected cases in the number remains unknown in general, which motivates the very study.

In this article, we generalize our previous works [3, 4, 5] on the topological spread model (and respectively, the random spread model) to the case where the matrix of substitution (respectively, the offspring mean matrix) is not necessarily primitive, and study the spread rate when individuals of a particular type are forced to give birth only to offspring of the same type. More precisely, this paper aims to characterize the spread rates of general spread models and to explain the periodic behavior of the spread rate that was discovered in the previous work [5].

1.2 Setup for the topological spread model with a frozen symbol

The setup for the topological model without a frozen symbol is presented, and we follow the notation of [5] for the reader's convenience.

Let $\mathcal{A} = \{a_i\}_{i=1}^{K}$ be a type set, and $\mathcal{T}_d$ be the conventional d-tree for $d \in \mathbb{N}$ with the root ϵ. Define $\Sigma^s = \{g \in \mathcal{T}_d : |g| = s\}$ for $s \in \mathbb{N}$ and $\Delta_n(h) = \{g \in \mathcal{T}_d :$ g is a descendant of h with $|g - h| < n\}$, where $|g-h|$ stands for the *length* of the unique path from h to g and $|g| = |g - \epsilon|$. We simply write $\Delta_n(\epsilon) = \Delta_n = \cup_{i=0}^{n-1}\Sigma^i$ for $h = \epsilon$. Denote $\Delta_m^n = \Delta_n \backslash \Delta_m = \{g \in \mathcal{T}_d : m \leq |g| < n\}$, and for $F \subseteq \mathcal{T}_d$ we define $F_m^n = F \cap \Delta_m^n$. Let $F \subseteq \Delta_1$, a function $p : F \to \mathcal{A}$ is called an 1-*pattern* and $F = F_p$ is called the *support* of p. Let $\mathcal{P}_1$ be the collection of all 1-patterns, and for $p \in \mathcal{P}_1$, we write $p^{(0)} = p(\epsilon) \in \mathcal{A}$ and for $g_1, \ldots, g_{d_p} \in F_p$ with $|g| = 1$, $d_p \in \mathbb{N}$, we write $p^{(1)} = (p(g_1), \ldots, p(g_{d_p}))$. Therefore, the 1-pattern p (see Figure 1) may also be stated as follows.

$$\begin{aligned} p &= (p^{(0)}, p^{(1)}) \\ &= (p(\epsilon); p(g_1), \ldots, g(g_{d_p^{(1)}})). \end{aligned}$$

Let $\mathcal{S} = \{p_i\}_{i=1}^{L} \subseteq \mathcal{P}_1$ and set $d = \max_{p \in \mathcal{S}} d_p$. The corresponding d-tree $\mathcal{T}_d$ is

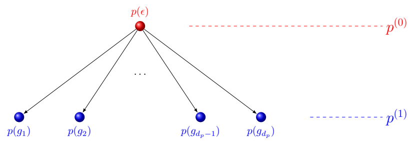


Figure 1: pattern

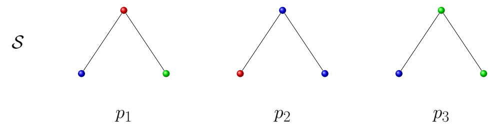


Figure 2: spread model

specified in the preceding paragraph, and other notations, e,g., Δ_m^n, F_m^n, etc., are also defined. The set $\mathcal{S}$ is called a *spread model* if $\forall p \in \mathcal{S}$ and $\forall g \in F_p$ with $|g| = 1$, then there exists a unique $q \in \mathcal{S}$ such that $q^{(0)} = p(g)$. See Figure 2 for an example of a spread model with 3 types.

Given a 1-spread model $\mathcal{S}$ and $p \in \mathcal{S}$, we define τ_p^∞ as follows, for which the process is illustrated in Figure 3. Let $\tau_p^0 = p^{(0)}$ and $\tau_p^1 = p$, for $g \in F_p$ with $|g| = 1$, since $\mathcal{S}$ is a 1-spread model, there exists a unique $q_g \in \mathcal{S}$ with $q_g^{(0)} = q_g(\epsilon) = p(g)$. As a result, we replace $p(g)$ by the 1-pattern q_g $\forall g \in F_p$ with $|g| = 1$ to generate a pattern τ_p^2. After τ_p^n is built, we substitute the pattern q_g for the symbol $\tau_p^n(g)$, $g \in F_{\tau_p^n}$ with $|g| = n$, to generate τ_p^{n+1}. Lastly, we define $\tau_p = \tau_p^\infty = \lim_{n\to\infty} \tau_p^n$ and call it the *infinite spread pattern induced from* p *with respect to* $\mathcal{S}$ (*induced spread pattern from* p). Given τ_p for some $p \in \mathcal{S}$ and $p^{(0)} = p(\epsilon) = b \in \mathcal{A}$, suppose $F \subset F_{\tau_p}$ is a finite set, we denote by $\tau_p|_F$ the subpattern of τ_p along the subset F, that is, $\tau_p|_F = \{\tau_p(g) : g \in F\}$. Given a sequence $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$, the following value $s_b(a; \mathcal{S}, \{k_n\}_{n=1}^\infty)$ is of interest and significance for the spread model $\mathcal{S}$.

$$s_b(a; \mathcal{S}, \{k_n\}_{n=1}^\infty) = \lim_{n\to\infty} s_b(s; [s_n, s_{n+1}]) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|},\ a \in \mathcal{A},$$

where $s_n = \sum_{i=1}^n k_i$ and $O_a(\tau_p|_F)$ is the number of occurrences of the type a in the range F.

Now we consider a 1-spread model $\mathcal{S} = \{p_i\}_{i=1}^L$ and introduce its induced spread models with a frozen symbol. Pick $f \in \mathcal{A}$ and $p_f \in \mathcal{S}$ so that $p_f^{(0)} = f$ (which is unique since $\mathcal{S}$ is a spread model). The pattern p_f is changed as follows.

$$\bar{p}_f = (\bar{p}_f^{(0)}; \bar{p}_f^{(1)}) = (f; f).$$

Define $\mathcal{S}^f = (\mathcal{S} \backslash \{p_f\}) \cup \{\bar{p}_f\}$ and call it the 1-*spread model with frozen symbol* f (see Figure 4 for an example). Given $\mathcal{S}^f$ and the induced spread pattern from

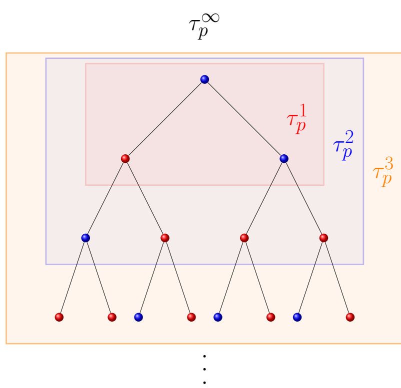


Figure 3: induced spread pattern

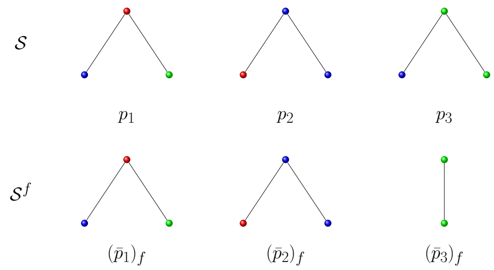


Figure 4: Spread model

p, say τ_p^f. Given a sequence $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$ and $s_n = \sum_{i=1}^n k_i$, the objective of this study is to calculate the spread rate

$$s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^\infty) = s_b(a; \mathcal{S}^f, \{k_n\}_{n=1}^\infty).$$

1.2.1 Previous results for topological spread models with a frozen symbol

Let $f \in \mathcal{A}$ be a frozen symbol, $\mathcal{S}^f$ be the 1-spread model and $f \in \mathcal{A}$ be a frozen symbol. The corresponding substitution map is denoted by ξ^f. Denote by $\mathbf{M}_0 = M_\xi$ and $\mathbf{M} = M_{\xi^f}$ the associated ξ-matrix and ξ^f-matrix respectively (The ξ-matrix and ξ^f-matrix are defined in Section 2). Since $\mathbf{M}_0$ is a $K \times K$ matrix, we denote by $G_{\mathbf{M}_0} = (V_{\mathbf{M}_0}, E_{\mathbf{M}_0})$ the associated graph of $\mathbf{M}_0$, that is, $V_{\mathbf{M}_0} = \mathcal{A}$, and $E_{\mathbf{M}_0} = \{(a_i, a_j) \in V_{\mathbf{M}_0} \times V_{\mathbf{M}_0} : \mathbf{M}_0(a_i, a_j) > 0\}$. We can readily verify that $\mathbf{M}$ is derived from $\mathbf{M}_0$ by deleting all edges with the property $(f, a) \in E_{\mathbf{M}_0}$. Therefore, we obtain

$$\mathbf{M} = \begin{bmatrix} M & 0 \\ \mathbf{C} & 1 \end{bmatrix}, \tag{1}$$

where $\mathbf{C} \in \mathbb{R}^{1 \times (K-1)}$ and $M \in \mathbb{R}^{(K-1)\times(K-1)}$. We stress that two additional hypotheses about the complexity of the spreading are made as in [5], namely, $\lim_{n\to\infty} |\xi(\alpha)| = \infty$ for all $\alpha \in \mathcal{A}$ and there exists $\xi(\alpha_0)$ beginning with α_0. Both conditions ensure that the associated ξ-matrix $\mathbf{M}_0$ is primitive, that is, $(\mathbf{M}_0)^k > 0$ for some $k \in \mathbb{N}$. However, the matrix $\mathbf{M}$ defined in (1) is not necessary primitive. Hence, we suppose

$$M = \begin{bmatrix} A^{[11]} & 0 & 0 & 0 \\ A^{[21]} & A^{[22]} & 0 & 0 \\ \vdots & \cdots & \ddots & 0 \\ A^{[r1]} & A^{[r2]} & \cdots & A^{[rr]} \end{bmatrix} \tag{2}$$

is lower triangular according to the irreducible decomposition of $\mathbf{M}$. Here and subsequently, we use $a \in V_A$ to denote the symbol a is in the index set of the matrix A.

It is surprising that the explicit value of the spread rate $s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^{\infty})$ depends heavily on the choice of the 'initial type' b and the 'target type' a. For $r = 2$, the values of $s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^{\infty})$ for $(a, b) \in \mathcal{A} \times \mathcal{A}$ are characterized in (Theorem 7, [5]), and we provide an algorithm for characterizing the explicit values of $s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^{\infty})$ for $r \geq 2$ (Theorem 11, [5]) as well. However, these results are far from being conclusive as two major conditions are imposed, namely, (1). $1 < \rho(A^{[ii]}) \neq \rho(A^{[jj]})$ for all $1 \leq i \neq j \leq r$, where $\rho(A^{[ii]})$ is the unique eigenvalue of $A^{[ii]}$ with $|\rho(A^{[ii]})| > |\lambda|$ for any other eigenvalues λ of $A^{[ii]}$ and $i \geq 1$ and (2). Each $A^{[ii]}$ is primitive for all $1 \leq i \leq r$. Conditions (1) and (2) appear not to be abandoned in the proofs of Theorem 7 and Theorem 11 of [5]. Our primary result in this article is to give a complete characterization of the exact values of $s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^{\infty})$ for $(a, b) \in \mathcal{A} \times \mathcal{A}$ without condition (1) or (2). Define $s_b^f(a) := s_b^f(a; \mathcal{S}, \{k_n\}_{n=1}^{\infty})$ with $k_n = 1 \ \forall \ n \geq 1$ for all $a, b \in \mathcal{A}$. In section 3, we prove the limit $s_b^f(a)$ exists and construct an algorithm to confirm $s_b^f(a) > 0$ (Theorem 3.1 and Theorem 3.2). Second, in comparison with the topological spread model, we provide a random version of the spread model with a frozen symbol. Such a model gives a more realistic picture of the spread phenomena in the real world. The random spread model with a frozen symbol will be presented in Section 1.3.

1.3 Random spread model with a frozen symbol

To introduce a spread model using the branching processes, we consider a population which starts with one individual and consists of individuals of K different types, say $a_1, a_2, \cdots, a_K$. Let

$$\mathbf{Z}_n = (Z_{n,1}, Z_{n,2}, \cdots, Z_{n,K})$$

be the population vector in the nth generation, where $Z_{n,i}$ is the number of individuals of type a_i in the nth generation, $i = 1, 2, \cdots, K$. Assume that each individual in the population lives a unit of time and, upon its death, produces its offspring independent of others in the same generation and in the past of the population. Assume that the production mechanism of each individual follows the probability distribution $\{p^{(i)}(\cdot)\}_{i=1}^K$, where $p^{(j)}(j_1, j_2, \cdots, j_K)$ is the probability that an individual of type a_i produces j_1 children of type a_1, j_2 children of type a_2, $\cdots$, and j_K children of type a_K. Then the process $\{\mathbf{Z}_n\}_{n\geq 0}$ is called a *K-type branching process with offspring distribution* $\{p^{(i)}(\cdot)\}_{i=1}^K$.

Let $m_{ji} = E(Z_{1,j}|\mathbf{Z}_0 = \mathbf{e}_i)$ be the expected value of the number of children of type a_j produced by an individual of type a_i, where $\mathbf{e}_i$ is the unit vector with 1 as its ith component. Then the matrix

$$\mathbf{M}_0 \equiv [m_{ji}] = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1K} \\ m_{21} & m_{22} & \cdots & m_{2K} \\ \vdots & \vdots & & \vdots \\ m_{K1} & m_{K2} & \cdots & m_{KK} \end{bmatrix}$$

is called the *offspring mean matrix* for this branching process $\{\mathbf{Z}_n\}_{n\geq 0}$. Moreover, if $\mathbf{M}_0^2 = \mathbf{M}_0 \cdot \mathbf{M}_0$ and $\mathbf{M}_0^n = \mathbf{M}_0^{n-1} \cdot \mathbf{M}_0$ for all $n \geq 3$, then the (j,i)-entry $m_{j,i}^{(n)}$ of the matrix $\mathbf{M}_0^n$ is the expected value

$$m_{ji}^{(n)} = E(Z_{n,j}|\mathbf{Z}_0 = \mathbf{e}_i)$$

of the number of offspring of type a_j in the nth generation of the population initiated by an ancestor of type a_i.

The behavior of the offspring matrix can provide the information about the branching process in the long run. In the theory of branching processes, a classical theorem tells us that, when the branching process is non-singular and the offspring mean matrix $\mathbf{M}_0$ is primitive with a maximal eigenvalue $\rho(\mathbf{M}_0) > 1$, the population vector converges geometrically almost surely:

$$\lim_{n\to\infty} \frac{\mathbf{Z}_n}{\rho(\mathbf{M}_0)^n} = \mathbf{v}_{\mathbf{M}_0}^t W \qquad a.s.$$

where $\mathbf{v}_{\mathbf{M}_0}$ is the normalized right eigenvector associated with $\rho(\mathbf{M}_0)$ and W is a random variable. We refer readers to Athreya and Ney [2] for more details.

It is known, from Ban et al. [3], that each multitype branching process induces a random spread model. Therefore, we assume that, for example, the spreading of certain kind of viruses can be modeled by a K-type branching process $\{\mathbf{Z}_n\}_{n\geq 0}$ with offspring distribution $\{p^{(i)}(\cdot)\}_{i=1}^K$, $\mathbf{Z_0} = \mathbf{e}_l$ and the offspring mean matrix $\mathbf{M}_0$. That is, individuals of different types in this population can be considered as patients in different categories such as highly contagious, moderately contagious, mildly contagious and so on. At some point, some epidemic prevention measures are applied to decrease the spread of this infectious disease. For instance, people who are tested and found to be highly contagious may be put in quarantine so that they will not pass the virus to others. Therefore, the spread pattern of this type changes and a random spread model with frozen symbols can be used to model this change. First of all, the category in which people are put in quarantine is labeled as type a_K for convenience and individuals in this category change their spread pattern so that, in branching language, each only produces exactly one "child" of the same type with probability one. Note that, to avoid the trivial case, we also assume that $l \neq K$. Therefore, this type a_K is considered to be "frozen" or "stopped" and a_K is called a *frozen type* or a *frozen symbol*. The change of the spread pattern of type a_K makes a difference in the branching mechanism and therefore affects the spread rate. Define the probability distribution $\{p^{*(i)}(\cdot)\}_{i=1}^K$ to be the modified offspring distribution after the type a_K is frozen as follows: $p^{*(i)}(j_1, j_2, \cdots, j_K) = p^{(i)}(j_1, j_2, \cdots, j_K)$ for all $i = 1, 2, \cdots, K-1$ and all $j_1, \cdots, j_K \in \mathbb{N}_0$ and

$$p^{*(K)}(j_1, j_2, \cdots, j_K) = \begin{cases} 1, & \text{if } (j_1, \cdots, j_K) = e_K; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{\mathbf{Z}_n^* = (Z_{n,1}^*, \cdots, Z_{n,K}^*)\}_{n\geq 0}$ be the branching process initiated with $\mathbf{Z_0^*} = \mathbf{e}_l$ and having $\{p^{*(i)}(\cdot)\}_{i=1}^K$ as its offspring distribution. Then, $\{\mathbf{Z}_n^*\}_{n\geq 0}$ is called the *associated (modified) branching process with frozen symbol a_K for the original branching process $\{\mathbf{Z}_n\}_{n\geq 0}$*. Let

$$m_{ji}^* \equiv E(Z_{1,j}^* | \mathbf{Z}_0^* = \mathbf{e}_i) = m_{ji}$$

for all $j = 1, 2, \cdots, K$ and $i = 1, 2, \cdots, K-1$ and let

$$m_{jK}^* \equiv E(Z_{1,j}^* | \mathbf{Z}_0^* = e_K) = \begin{cases} 1, & \text{if } j = K; \\ 0, & \text{if } j = 1, 2, \cdots, K-1. \end{cases}$$

Then the offspring mean matrix for the process $\{\mathbf{Z}_n^*\}_{n\geq 0}$ is given by

$$\mathbf{M} \equiv [m_{ji}^*].$$

In such a random model initiated with an individual of type a_l, we define the *spread rate of the type* a_j as follows:

$$s^f_{a_l}(a_j) = \lim_{n\to\infty} \frac{Z^*_{n,j}}{\sum\limits_{j=1}^{K} Z^*_{n,j}}$$

where $l, j = 1, 2, \cdots, K$. Note that the spread rate $s^f_{a_l}(a_j)$ is a random quantity, and so the first question that may arise is about its existence as a limit of a sequence of the random variables. If so, in what sense does the limit converge? The classical results in the theory of branching processes already give the answer when the offspring mean matrix is primitive. However, it is obvious that the offspring mean matrix $\mathbf{M}$ for the associated branching process $\{\mathbf{Z}^*_n\}_{n\geq 0}$ with frozen symbol a_K is no longer primitive (However, throughout this paper, we keep the non-singularity for the associated process with frozen symbol to avoid the trivial cases in which each individual has exactly one offspring.) and therefore the classical convergence theorems can not be applied to this modified process directly to find the spread rate of each type. In Section 4, we will show that the almost sure convergence of the spread rate $s^f_{a_l}(a_j)$, for all $l, j = 1, 2, \cdots, K$, still holds for a more general offspring mean matrix $\mathbf{M}$ with some conditions on its covariance matrix.

2 Preliminaries

2.1 Spread rate

Let $\mathcal{A} = \{a_i\}_{i=1}^K$ be a type set and $\mathcal{A}^* = \cup_{n=0}^{\infty} \mathcal{A}^n$, where $\mathcal{A}^n$ is the set of all n-blocks, i.e., the blocks with length $n \in \mathbb{N}$. Assume $\mathcal{S}$ is a 1-spread model with $p = (p^{(0)}; p^{(1)}) = (b; p^{(1)}) \in \mathcal{S}$. We called the map $\xi : \mathcal{A} \to \mathcal{A}^*$ associated *substitution* on $\mathcal{A}$ if $\xi(b) = p^1$ with length $|\xi(b)| = d_p^{(1)}$, where $d_p^{(1)}$ is the number of $g \in F_p$ with $|g| = 1$. To keep the notation simple, we defined $\xi(w) = \xi(w_1)\cdots\xi(w_n)$ for all $w = w_1 \cdots w_n \in \mathcal{A}^*$ and $\xi(\emptyset) = \emptyset$. The associated ξ-*matrix* $M_\xi := [O_{a_i}(\xi(a_j))]$ is a $K \times K$ matrix, where $O_{a_i}(\xi(a_j))$ is the number of a_i appearing in the $\xi(a_j)$. Meanwhile, $L(w) = (O_{a_1}(w), O_{a_2}(w), ..., O_{a_k}(w))$ for all $w \in \mathcal{A}^*$. Let M be the irreducible decomposition of M_ξ which is defined in (2).

In Section 3, we will calculate the value $s^f_b(a)$. More precisely, for all $p \in \mathcal{S}$

with $p^{(0)} = b$, $a = a_j$ and $b = a_i$, we have

$$s_b^f(a) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Sigma^n(\tau_p)})}{|\Sigma^n(\tau_p)|} = \lim_{n\to\infty} \frac{\mathbf{e}_j^t M^n \mathbf{e}_i}{\mathbf{1}_K M^n \mathbf{e}_i}, \tag{3}$$

where $\mathbf{1}_K$ is a $1 \times K$ vector with all its entries all 1.

2.2 Notations and definitions for matrices

For convenience and to simplify the proof in this article, we denote

$$M_k = \begin{bmatrix} A^{[kk]} & 0 & 0 \\ \vdots & \ddots & 0 \\ A^{[rk]} & \dots & A^{[rr]} \end{bmatrix} \text{ and } M^n = \begin{bmatrix} A_n^{[11]} & 0 & 0 & 0 \\ A_n^{[21]} & A_n^{[22]} & 0 & 0 \\ \vdots & \dots & \ddots & 0 \\ A_n^{[r1]} & \dots & \dots & A_n^{[rr]} \end{bmatrix}.$$

Let A be a $d \times d$ matrix. An eigenvalue of A is called *maximal* if it is the unique eigenvalue $\rho(A)$ of A with $|\rho(A)| > |\lambda|$ for any other eigenvalues λ of A. The right eigenvector of the $\rho(A)$, say $\mathbf{v}_A$ (resp. $\mathbf{w}_A$), is called the *right (resp. left) maximal eigenvector* of A. In particular, for the matrix M defined in (1) and (2), we write $\rho = \rho(M)$, $\mathbf{v} = \mathbf{v}_M$ and $\mathbf{w} = \mathbf{w}_M$.

Suppose B is another $d \times d$ matrix, we say $A \leq B$ if $A_{i,j} \leq B_{i,j}$ for all $1 \leq i, j \leq d$. Additionally, we say A is *primitive* if there exists $n \in \mathbb{N}$ such that $A^n_{i,j} > 0$ for all $1 \leq i, j \leq d$. A matrix A is called *irreducible* if for all $1 \leq i, j \leq d$ there exists $n \in \mathbb{N}$ such that $A^n_{i,j} > 0$. A square matrix A satisfying $A^n = A$ is called *n-periodic*.

3 Spread rate for topological spread model

In this section, we first establish the existence of the limit of $s_b^f(a)$. To do this, we consider different cases and introduce an algorithm to determine whether $s_b^f(a) > 0$ or not. Theorem 3.1 addresses the case where $A^{[ii]}$ is primitive for all $i = 1, ..., r$, while Theorem 3.2 considers the case where $A^{[ii]}$ is irreducible. Both results provide a complete characterization of the positive spread rate. Finally, we present the algorithm following Propositions 3.5 and 3.7.

The positivity of the spread rate explains the long-term behavior of the spread of each type. Namely, when the spread rate of a certain type is positive, the symbols or the individuals of this type survive in a long run. Therefore, this characteristic is a crucial aspect of our study. Theorem 3.1 and 3.2 provide the general structure of a set that guarantees a positive spread rate. In Section 5, we give a specific method for identifying the elements of this set.

Theorem 3.1. *Suppose $A^{[ii]}$ is primitive for all $i = 1, ..., r$. Assume $b \in V_{A^{[kk]}}$ for some $1 \leq k \leq r$. There exists a set $\mathcal{P}_b$ with respect to b such that*

(i) if $a \in V_{A^{[pp]}}$ for all $p \in \mathcal{P}_b$, then $s_b^f(a) > 0$.

(ii) if $s_b^f(a) > 0$, then $a \in V_{A^{[pp]}}$ for some $p \in \mathcal{P}_b$.

Theorem 3.2. *Suppose $A^{[ii]}$ is irreducible with period p_i respectively for $i = 1, ..., r$. Assume $b \in V_{A^{[kk]}}$ for some $1 \leq k \leq r$. Along the subsequence $np + z$, where $z = 0, ..., p-1$ and $p =$lcm$\{p_k, ..., p_r\}$, there exists a set $\mathcal{P}_{b;z}$ with respect to b and z such that*

(i) if $a \in V_{A^{[qq]}}$ for all $q \in \mathcal{P}_{b;z}$, then $s_b^f(a) > 0$.

(ii) if $s_b^f(a) > 0$, then $a \in V_{A^{[qq]}}$ for some $q \in \mathcal{P}_{b;z}$.

Remark 3.3. *Our goal is to determine the situations in which $s_b^f(a) > 0$. It is important to note that by applying the same idea as described in this section, the actual value of $s_b^f(a)$ in Theorem 3.1 and Theorem 3.2 can be calculated ((4) and (5)). Let $\mathbf{w}_{i,j}^t$ (resp. $\mathbf{v}_{ij}$) be the left (resp. right) generalized eigenvectors associated with $\rho(M) > 0$ which lie in the ith maximal Jordan block for $i = 1, ..., d$ such that $\mathbf{w}_{i,j}^t M = \rho(M)\mathbf{w}_{i,j} + \mathbf{w}_{i,j-1}$, where $j = 1, ..., N$ and $\mathbf{w}_{i,0} = 0$. Suppose the assumption in Theorem 3.1 holds, we obtain*

$$s_b^f(a) = \frac{\sum_{\ell=1}^d \mathbf{w}_{\ell,k}(b)\mathbf{v}_{\ell,d}(a)}{\sum_{c \in \cup_{h=k}^r V_{A^{[hh]}}} \sum_{\ell=1}^d \mathbf{w}_{\ell,k}(b)\mathbf{v}_{\ell,d}(c)}, \tag{4}$$

where $\mathbf{w}_{\ell,k}(b)$ (resp. $\mathbf{v}_{\ell,d}(a)$) is the entry indexed by b (resp a) and $a \in \cup_{h=k}^r V_{A^{[hh]}}$. Under the assumption in Theorem 3.2, we obtain the value

$$s_b^f(a) = \frac{\sum_{\ell=1}^d \mathbf{w}_{\ell,k}(b)(M^z\mathbf{v}_{\ell,d})(a)}{\sum_{c \in \cup_{h=k}^r V_{A^{[hh]}}} \sum_{\ell=1}^d \mathbf{w}_{\ell,k}(b)(M^z\mathbf{v}_{\ell,d})(c)}. \tag{5}$$

Before proving Theorem 3.1 and Theorem 3.2, we elaborate on how to find $\mathcal{P}_b$ in Theorem 3.1 (resp. $\mathcal{P}_{b;z}$ in Theorem 3.2) first. Without loss of generality, we make a standing assumption that $b \in V^{[11]}$, since otherwise, one can construct an index set

$$\mathscr{I}_k = \{k \le i \le r : A_n^{[ik]} \ne 0 \text{ for some } n \in \mathbb{N}\},$$

and a submatrix M' of M that is defined as

$$M' = [A^{[ij]}]_{ij \in \mathscr{I}_k^2 : i > j}.$$

This submatrix contains all the offspring information contributed by ancestors of type b in the sense that

$$\mathbf{e}_a^t M^n \mathbf{e}_b = \begin{cases} \mathbf{e}_a^t {M'}^n \mathbf{e}_b & \text{if } a \in \cup_{n \in \mathscr{I}_k} V^{[nn]} \\ 0 & \text{otherwise.} \end{cases}$$

We first present the following fundamental lemma, which is frequently used in the development of our theory.

Lemma 3.4. *Suppose A is a $d \times d$ non-negative irreducible matrix with period p. Then, there exist some constants $c > 0$, $1 > \gamma > 0$, and non-zero matrices $\{B_z : 0 \le z < p\}$ such that $|\frac{A^{mp+z}}{\rho^{mp+z}} - B_z| \le c\gamma^{mp+z}$ for all $m \ge 0$.*

Proof. If A is primitive, or equivalently, $p = 1$, then we consider the Jordan form of

$$A = \begin{bmatrix} V_1 & \cdots & V_m \end{bmatrix} \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_m \end{bmatrix} \begin{bmatrix} U_1^t \\ \vdots \\ U_m^t \end{bmatrix} = \sum_{i=1}^m V_i J_i U_i^t,$$

and thus $A^n = \sum_{i=1}^m V_i J_i^n U_i$, where we assume $|\rho(J_1)| \ge |\rho(J_2)| \ge \cdots \ge |\rho(J_m)|$. Now that A is primitive, we have $\rho(A) = |\rho(J_1)| > |\rho(J_2)|$, from which the lemma follows immediately by taking $1 > \gamma > |\rho(J_2)|/|\rho(J_1)|$. If $p > 1$, A^p is a diagonal block matrix with each block on the diagonal a primitive matrix. We are then able to apply the case $p = 1$ to the diagonal blocks of A^p to obtain the desired estimate. This proof is complete. □

Let $\mathscr{S} := \{i : \rho(A^{[ii]}) = \rho(M)\}$. For each $1 \le i \le k$, define

$$S_i := \{\mathtt{i} = \mathtt{i}_1 \cdots \mathtt{i}_\ell : \mathtt{i}_1 = 1, \mathtt{i}_\ell = i, \mathtt{i}_j \text{ are distinct s.t. } A^{[\mathtt{i}_{j+1}\mathtt{i}_j]} \ne 0 \ \forall j\},$$

$$N := \max \#\{\ell : \mathtt{i} = \mathtt{i}_1 \cdots \mathtt{i}_\ell \in S_i\},$$

$$S_i' := \{\mathtt{i} \in S_i : \#\{j : \mathtt{i}_j \in \mathscr{S}\} = N\} \text{ and } S_i'' := S_i \setminus S_i'.$$

In addition, for any $\ell, L, n \in \mathbb{Z}_+$, let

$$\Omega_{\ell,n} := \left\{ (n_1, \cdots, n_\ell) \in \mathbb{Z}_+ : \sum_{i=1}^{\ell} n_i = n - \ell + 1 \right\},$$

$$\Omega_{\ell,n,L} := \{(n_1, \cdots, n_\ell) \in \Omega_{\ell,n} : n_i \geq L, \forall i\}.$$

It is not hard to see that

$$\frac{A_n^{[i1]}}{\binom{n}{N-1}\rho^{n-N+1}} = \sum_{\mathtt{i}\in S_i} \frac{1}{\binom{n}{N-1}} \sum_{(n_1,\cdots,n_{|\mathtt{i}|})\in\Omega_{|\mathtt{i}|,n}} \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}}. \tag{6}$$

Proposition 3.5. *Suppose each block on the diagonal of lower triangular block matrix M is either primitive or zero. Then, the limit $\lim_{n\to\infty} \frac{A_n^{[i1]}}{\binom{n}{N-1}\rho^{n-N+1}}$ exists, and it is non-zero if and only if it is positive if and only if S_i' is nonempty.*

Proof. Let ρ be the spectral radius of M. Suppose $B^{[jj]} = \lim_{m\to\infty} \frac{A_n^{[jj]}}{\rho^n}$ is as found in Lemma 3.4, for which we may find constants $c > 1$ and $1 > \gamma > 0$ such that $|A^{[jj']}|, |B^{[jj']}| \leq c$ for all j, j' and that $\left|\frac{A_n^{[jj]}}{\rho^n} - B^{[jj]}\right| \leq c\gamma^n$. In the following, we divide our discussion, in terms of $\mathtt{i}$, into two cases and prove respectively the convergence of the summand in (6).

Case 1: $\mathtt{i} \in S_i''$. We show that the summand of (6) converges to 0 as $n \to \infty$. Let $\Lambda_{n,L}(\mathtt{i}) = \{(n_1, \cdots, n_{|\mathtt{i}|}) \in \Omega_{|\mathtt{i}|,n} : \sum_{(j:\mathtt{i}_j\notin\mathscr{S})} n_j = L\}$ and $\alpha = \#\{j : \mathtt{i}_j \in \mathscr{S}\} \leq N - 1$. Then, for all $L \neq 0$,

$$\#\Lambda_{n,L}(\mathtt{i}) = \binom{L + |\mathtt{i}| - \alpha - 1}{|\mathtt{i}| - \alpha - 1}\binom{n - L - |\mathtt{i}| + \alpha}{\alpha - 1} \leq |\mathtt{i}|^{|\mathtt{i}|} L^{|\mathtt{i}|} n^{\alpha-1},$$

and thus we have the following estimate for all $\mathtt{i} \in S_i$ and sufficiently large n:

$$\begin{aligned}
&\frac{1}{\binom{n}{N-1}} \sum_{(n_1,\cdots,n_{|\mathtt{i}|})\in\Omega_{|\mathtt{i}|,n}} \left| \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}} \right| \\
=&\frac{1}{\binom{n}{N-1}} \sum_{L=0}^{n-|\mathtt{i}|+1} \sum_{(n_1,\cdots,n_{|\mathtt{i}|})\in\Lambda_{n,L}(\mathtt{i})} \left| \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}} \right| \\
\leq&\frac{2^\alpha}{n^\alpha} \sum_{L=0}^{n-|\mathtt{i}|+1} \#\Lambda_{n,L}(\mathtt{i}) \cdot c^{|\mathtt{i}|}\gamma^L \leq \frac{(2c|\mathtt{i}|)^{|\mathtt{i}|}}{n} \left(1 + \sum_{L=1}^{n-|\mathtt{i}|+1} L^{|\mathtt{i}|}\gamma^L\right).
\end{aligned}$$

Since it is readily checked that $\sum_{L=1}^{\infty} L^{|\mathtt{i}|}\gamma^L < \infty$, the claim is proved.

Case 2: $\mathtt{i} \in S_i'$**.** We claim that the summand in (6) converges to

$$B_*^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} B_*^{[11]},$$

where

$$B_*^{[\mathtt{i}_j\mathtt{i}_j]} = \begin{cases} B^{[\mathtt{i}_j\mathtt{i}_j]}, & \text{if } \mathtt{i}_j \in \mathscr{S} \\ \sum_{\ell=0}^{\infty} \rho^{-\ell} A_\ell^{[\mathtt{i}_j\mathtt{i}_j]} = (I - \rho^{-1}A^{[\mathtt{i}_j\mathtt{i}_j]})^{-1} & \text{if } \mathtt{i}_j \notin \mathscr{S}. \end{cases}$$

To prove this, define for each $n \in \mathbb{Z}_+$

$$C_n^{[\mathtt{i}_j\mathtt{i}_j]} = \begin{cases} B^{[\mathtt{i}_j\mathtt{i}_j]} & \text{if } \mathtt{i}_j \in \mathscr{S}; \\ \rho^{-n} A_n^{[\mathtt{i}_j\mathtt{i}_j]} & \text{if } \mathtt{i}_j \notin \mathscr{S}, \end{cases}$$

so that $|C_n^{[\mathtt{i}_j\mathtt{i}_j]}| \le c$. Hence, the summand of (6) can be written as

$$\begin{aligned} &\frac{1}{\binom{n}{N-1}} \sum_{(n_1,\cdots,n_{|\mathtt{i}|}) \in \Omega_{|\mathtt{i}|,n}} \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}} \\ =& \sum_{[(n_j)_{\mathtt{i}_j \notin \mathscr{S}}] \in \mathbb{Z}_+^{|\mathtt{i}|-N}} \frac{1}{\binom{n}{N-1}} \sum_{\substack{[(n_j)_{\mathtt{i}_j \in \mathscr{S}}] \in \\ \Omega_{N,n-\sum_{(j:\mathtt{i}_j \notin \mathscr{S})} n_j}}} \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}}, \end{aligned} \tag{7}$$

whose limit could then be obtained by applying the dominated convergence theorem. More specifically, given any $(n_j)_{\mathtt{i}_j \notin \mathscr{S}}$,

$$\frac{1}{\binom{n}{N-1}} \sum_{\substack{[(n_j)_{\mathtt{i}_j \in \mathscr{S}}] \in \\ \Omega_{N,n-\sum_{(j:\mathtt{i}_j \notin \mathscr{S})} n_j}}} \left| \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}} \right| \le c^{|\mathtt{i}|} \gamma^{\sum_{(j:\mathtt{i}_j \notin \mathscr{S})} n_j}.$$

On the other hand, each term on the right-hand side of (7) converges to

$$C_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} C_{n_1}.$$

Indeed, since $\#[\Omega_{N,n-\sum_{(j:\mathtt{i}_j \notin \mathscr{S})} n_j}]/\binom{n}{N-1} \to 1$ as $n \to \infty$, we have

$$\begin{aligned} &\frac{1}{\#\Omega_{N,n}} \sum_{[(n_j)_{\mathtt{i}_j \in \mathscr{S}}] \in \Omega_{N,n}} \Bigg| \frac{A_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}}{\rho^{n_{|\mathtt{i}|}}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} \frac{A_{n_1}^{[11]}}{\rho^{n_1}} \\ &\qquad - C_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} C_{n_1}^{[11]} \Bigg| \\ \le& \frac{1}{\#\Omega_{N,n}} \left[\#\Omega_{|\mathtt{i}|-N,n,\sqrt{n}} \cdot |\mathtt{i}| \cdot (c^{|\mathtt{i}|}\gamma^{\sqrt{n}}) + \#(\Omega_{|\mathtt{i}|-N,n} \setminus \Omega_{|\mathtt{i}|-N,n,\sqrt{n}}) \cdot (2c) \right] \to 0. \end{aligned}$$

Our claim is then proved by the dominated convergence theorem.

Finally, from Case 2 one sees that the limit $\lim_{n\to\infty} \frac{A_n^{[i1]}}{\binom{n}{N-1}\rho^{n-N+1}}$ is non-zero if and only if $S'_i \neq \emptyset$. This proof is complete. □

Proof of Theorem 3.1. Let $\mathcal{P}_b = \{i : S'_i \neq \emptyset\}$ so that Proposition 3.5 immediately yields that $p \in \mathcal{P}_b$ if and only if

$$\lim_{n\to\infty} \frac{A_n^{[p1]}}{\binom{n}{N-1}\rho^{n-N+1}} > 0.$$

□

Now, it remains to show the case where diagonal elements are irreducible. To show this, we further divide each block $A^{[ij]}$ into sub-blocks $A^{[ij],[i'j']}$ such that $A_{p_i}^{[ii],[i'i']}$ is either primitive or zero, where p_i is the period of $A^{[ii]}$. We then have a derived block matrix (which is essentially the same as M) with its block denoted as ${A'}^{[ii]}$. In a similar manner, we define

$$\mathscr{S}^* := \{i : {A'}^{[ii]} = A^{[i'i'],[i''i'']}, \rho(A^{[i'i']}) = \rho(M)\}$$

and for each $1 \leq i \leq k$, define

$$S_i^* := \{\mathtt{i} = \mathtt{i}_1 \cdots \mathtt{i}_\ell : \mathtt{i}_1 = 1, \mathtt{i}_\ell = i, \mathtt{i}_j \text{ are distinct s.t. } {A'}^{[\mathtt{i}_{j+1}\mathtt{i}_j]} \neq 0 \ \forall j\}.$$

By denoting $\mathtt{i}_j \sim \mathtt{i}_{j'}$ if it corresponds to the same block matrix $A^{[ii]}$, we then define

$$S_i'^* := \{\mathtt{i} \in S_i^* : \#\{j : \mathtt{i}_j \in \mathscr{S}^*, \mathtt{i}_j \nsim \mathtt{i}_{j-1}\} = N\} \text{ and } S_i''^* := {S_i}^* \setminus S_i'^*.$$

We then begin a discussion with the following lemma. The greatest common divisor of a vector $\mathbf{p} = (p_1, ..., p_d) \in \mathbb{N}^d$ means the greatest common divisor of $p_1, ..., p_d$ and is denoted by $\gcd \mathbf{p}$. The inner product of two vectors $v, w \in \mathbb{Z}_+^d$ is denoted by $v \cdot w$.

Lemma 3.6. *Let* $\mathbf{p} \in \mathbb{N}^d$*,* $P = \gcd \mathbf{p}$*, and* $\gamma_{\mathbf{p},P;n} := \#\{\mathbf{n} \in \mathbb{Z}_+^d : \mathbf{p} \cdot \mathbf{n} = Pn\}/\binom{Pn}{d-1}$*. Then,* $\gamma_{\mathbf{p},P} := \lim_{n\to\infty} \gamma_{\mathbf{p},P;n} = \operatorname{lcm}\{\prod_{i\neq j} \mathbf{p}_i : 1 \leq j \leq d\}$*.*

Proof. The case $d = 1, 2$ is clear. The rest follows by induction. □

Proposition 3.7. *Suppose each block on the diagonal of a lower triangular block matrix M is either irreducible or zero, and $P = \operatorname{lcm}\{\gcd\{p_{\mathtt{i}_j} : \mathtt{i}_j \in \mathscr{S}\} : \mathtt{i} \in S_i'\}$. Then, the limit $\lim_{n\to\infty} \frac{A'^{[i1]}_{Pn+z}}{\binom{Pn+z}{N-1}\rho^{Pn+z-N+1}}$ exists for all $z \in \mathbb{Z}_+$, and is non-zero if and only if it is positive if and only if $S_i'^*$ is nonempty.*

Proof. Let ρ be the spectral radius of M and p_j be the period of $A^{[jj]}$. Suppose $B_z^{[jj]} = \lim_{n\to\infty} \frac{A^{[jj]}_{p_j n+z}}{\rho^{p_j n+z}}$ is as found in Lemma 3.4, for which we may find constants $c > 1$ and $1 > \gamma > 0$ such that $|A^{[jj']}|, |B_z^{[jj']}| \le c$ for all j, j', $z \in \mathbb{Z}_+$ and that $\left|\frac{A^{[jj]}_{p_j n+z}}{\rho^{p_j n+z}} - B_z^{[jj]}\right| \le c\gamma^n$. We consider in the following the same cases as those of Proposition 3.5.

Case 1: $\mathtt{i} \in S_i''$. The summand converges to 0 as $n \to \infty$, and the argument is exactly the same as that in Proposition 3.5.

Case 2: $\mathtt{i} \in S_i'$. Let $q_j = \operatorname{lcm}\{p_j, P\}$, and

$$B^{[\mathtt{i}_j\mathtt{i}_j]}_{*,z} = \begin{cases} B_z^{[\mathtt{i}_j\mathtt{i}_j]} & \text{if } \mathtt{i}_j \in \mathscr{S}; \\ \sum_{\ell=0}^{\infty} \rho^{-(q_{\mathtt{i}_j}\ell+z)} A^{[\mathtt{i}_j\mathtt{i}_j]}_{q_{\mathtt{i}_j}\ell+z} = \rho^{-z} A_z^{[\mathtt{i}_j\mathtt{i}_j]}(I - \rho^{-q_{\mathtt{i}_j}} A^{[\mathtt{i}_j\mathtt{i}_j]}_{q_{\mathtt{i}_j}})^{-1} & \text{if } \mathtt{i}_j \notin \mathscr{S}. \end{cases}$$

We show that the summand converges to

$$\sum_{\substack{(r_j)_{1\le j\le|\mathtt{i}|} \in \mathbb{Z}_+^{|\mathtt{i}|}: \\ 0 \le r_j < p_{\mathtt{i}_j} \text{ if } \mathtt{i}_j \in \mathscr{S}, \\ 0 \le r_j < q_{\mathtt{i}_j} \text{ if } \mathtt{i}_j \notin \mathscr{S}, \\ P | \left(|\mathtt{i}|-1+\sum_j r_j\right)}} \gamma_{P,(p_{\mathtt{i}_j})_{\mathtt{i}_j \notin \mathscr{S}}} B^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}_{*,r_{|\mathtt{i}|}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} B^{[11]}_{*,r_1},$$

where the notation $P \mid \left(|\mathtt{i}| - 1 + \sum_j r_j\right)$ means that P divides $|\mathtt{i}| - 1 + \sum_j r_j$.

To this end, define for each $n \in \mathbb{Z}_+$ and $0 \le z < p_{\mathtt{i}_j}$

$$C^{[\mathtt{i}_j\mathtt{i}_j]}_{p_{\mathtt{i}_j}n+z} = \begin{cases} B_z^{[\mathtt{i}_j\mathtt{i}_j]} & \text{if } \mathtt{i}_j \in \mathscr{S}; \\ \rho^{-(p_{\mathtt{i}_j}n+z)} A^{[\mathtt{i}_j\mathtt{i}_j]}_{p_{\mathtt{i}_j}n+z} & \text{if } \mathtt{i}_j \notin \mathscr{S}, \end{cases}$$

so that $|C^{[\mathtt{i}_j\mathtt{i}_j]}_{p_{\mathtt{i}_j}n+z}| \le c$. Due to the same argument as in Proposition 3.5, we know that the summand and the following sequence are equiconvergent:

$$\sum_{[(n_j)_{\mathtt{i}_j\notin\mathscr{S}}] \in \mathbb{Z}_+^{|\mathtt{i}|-N}} \frac{1}{\binom{n}{N-1}} \sum_{\substack{[(n_j)_{\mathtt{i}_j\in\mathscr{S}}] \in \\ \Omega_{N,\left(n-|\mathtt{i}|+N-\sum_{(j:\mathtt{i}_j\notin\mathscr{S})} n_j\right)}}} C^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]}_{n_{|\mathtt{i}|}} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]} \cdots A^{[\mathtt{i}_2\mathtt{i}_1]} C^{[11]}_{n_1}. \tag{8}$$

Now we consider a subsequence of (8) on Pn, for which by Lemma 3.6 we have

$$\lim_{n\to\infty} \sum_{[(n_j)_{\mathtt{i}_j\notin\mathscr{S}}]\in\mathbb{Z}_+^{|\mathtt{i}|-N}} \frac{1}{\binom{Pn}{N-1}} \sum_{\substack{[(n_j)_{\mathtt{i}_j\in\mathscr{S}}]\in \\ \Omega_{N,\left(Pn-|\mathtt{i}|+N-\sum_{(j:\mathtt{i}_j\notin\mathscr{S})} n_j\right)}}} C_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]}\cdots A^{[\mathtt{i}_2\mathtt{i}_1]} C_{n_1}^{[11]}$$

$$= \sum_{\substack{[(n_j)_{\mathtt{i}_j\notin\mathscr{S}}]\in\mathbb{Z}_+^{|\mathtt{i}|-N} \\ P|\left(|\mathtt{i}|-1+\sum_{(j:\mathtt{i}_j\notin\mathscr{S})} n_j\right)}} \sum_{\substack{[(n_j)_{\mathtt{i}_j\in\mathscr{S}}]\in\mathbb{Z}_+^{N}: \\ 0\le n_j<p_{\mathtt{i}_j}}} \gamma_{P,(p_{\mathtt{i}_j})_{\mathtt{i}_j\in\mathscr{S}}} C_{n_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]}\cdots A^{[\mathtt{i}_2\mathtt{i}_1]} C_{n_1}^{[11]}$$

$$= \sum_{\substack{[(r_j)_{1\le j\le|\mathtt{i}|}]\in\mathbb{Z}_+^{|\mathtt{i}|}: \\ 0\le r_j<p_{\mathtt{i}_j} \text{ if } \mathtt{i}_j\in\mathscr{S}, \\ 0\le r_j<q_{\mathtt{i}_j} \text{ if } \mathtt{i}_j\notin\mathscr{S}, \\ P|\left(|\mathtt{i}|-1+\sum_j r_j\right)}} \gamma_{P,(p_{\mathtt{i}_j})_{\mathtt{i}_j\in\mathscr{S}}} B_{*,r_{|\mathtt{i}|}}^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|}]} A^{[\mathtt{i}_{|\mathtt{i}|}\mathtt{i}_{|\mathtt{i}|-1}]}\cdots A^{[\mathtt{i}_2\mathtt{i}_1]} B_{*,r_1}^{[11]}.$$

Finally, it is not hard to verify the positivity from the above expression in Case 2. □

Proof of Theorem 3.2. Let $\mathcal{P}_{b;z} = \{i : S_i'^* \neq \emptyset\}$. Applying the Proposition 3.7, we obtain $q \in \mathcal{P}_{b;z}$ if and only if

$$\lim_{n\to\infty} \frac{A_{pn+z}^{[q1]}}{\binom{pn+z}{N-1}\rho^{pn+z-N+1}} > 0.$$

We can figure out the results of the theorem based on this fact. This proof is complete. □

4 Spread rate for random spread model with a frozen symbol

In Section 4.1, we start with the case where the offspring mean matrix contains exactly one primitive component. We present the almost surely convergence of geometrically normalized the population vector in Theorem 4.1 and find the direction of the limit of the normalized population vector in Theorem 4.2. These two results together allow us to investigate the spread rates for the random spread model in Theorem 4.3. We also state the results for the case where the offspring mean matrix has two primitive components in Theorem 4.5. Moreover, all the proofs of the main results and lemmas are provided in Section 4.2.

4.1 Main results on the spread rate in a random spread model with a frozen symbol

We first assume that the offspring mean matrix for the associated process $\{\mathbf{Z}_n^*\}_{n\geq 0}$ with a frozen type a_K can be represented as

$$\mathbf{M} \equiv [m_{ji}^*] = \begin{bmatrix} & & & 0 \\ & M & & \vdots \\ & & & 0 \\ m_{K1} & \cdots & m_{KK-1} & 1 \end{bmatrix}$$

where M is the matrix

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1K-1} \\ m_{21} & m_{22} & \cdots & m_{2K-1} \\ \vdots & \vdots & \cdots & \vdots \\ m_{K-11} & m_{K-12} & \cdots & m_{K-1K-1} \end{bmatrix}.$$

and let the covariance matrix $\mathbf{V}^{(l)}$ be defined as the following:

$$\mathbf{V}^{(l)} = Var(\mathbf{Z}_1^*|\mathbf{Z}_0^* = \mathbf{e}_l)$$

where its (i,j)-entry is

$$E(Z_{1,i}^* Z_{1,j}^*|\mathbf{Z}_0^* = \mathbf{e}_l) - E(Z_{1,i}^*|\mathbf{Z}_1^* = \mathbf{e}_l)E(Z_{1,j}^*|\mathbf{Z}_0^* = \mathbf{e}_l)$$

for all $i, j = 1, 2, \cdots, K$.

Theorem 4.1. *Let $\rho = \rho(\mathbf{M}) > 1$ be the maximal eigenvalue of the offspring mean matrix $\mathbf{M}$. If the matrix M is primitive and the covariance matrices $\mathbf{V}^{(l)}$'s are finite for all $l = 1, 2, \cdots, K$, then there exists a random vector $\mathbf{W} = (W_1, W_2, \cdots, W_K)$ with $E(\mathbf{W}^t\mathbf{W}) < \infty$ such that*

$$\frac{\mathbf{Z}_n^*}{\rho^n} \to \mathbf{W} \qquad \text{as } n \to \infty$$

both in mean square and with probability 1.

Let $\mathbf{w} = (w_1, w_2, \cdots, w_K)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_K)^t$ be the left and right eigenvector of $\mathbf{M}$ associated with the maximal eigenvalue ρ. It can seen from linear algebra that, if M is primitive, then $v_i > 0$ for all $i = 1, 2, \cdots, K$, $w_i > 0$ for all $i = 1, 2, \cdots, K-1$ and $w_K = 0$. So, we can normalize $\mathbf{v}$ and $\mathbf{w}$ such that

$$\sum_{i=1}^{K} v_i = 1 \quad \text{and} \quad \sum_{i=1}^{K} w_i v_i = 1.$$

Throughout this paper, when we deal with random spread models, $\mathbf{v}$ and $\mathbf{w}$ are assumed to be such normalized eigenvectors.

The next theorem tells us that the limit vector $\mathbf{W}$ has the same direction as the right eigenvector $\mathbf{v}$ of the offspring mean matrix $\mathbf{M}$ for the spread model:

Theorem 4.2. *Let $\rho = \rho(\mathbf{M}) > 1$ be the maximal eigenvalue of the offspring mean matrix $\mathbf{M}$ and let $\mathbf{w}$ and $\mathbf{v}$ be the normalized left and right eigenvectors of $\mathbf{M}$ associated with ρ. Let the matrix M be primitive, the covariance matrices $\mathbf{V}^{(l)}$'s be finite for all $l = 1, 2, \cdots, K$ and the random vector $\mathbf{W}$ be defined as in Theorem 4.1. Then there exists a random variable W such that*

$$\mathbf{W} = \mathbf{v}^t W$$

with probability 1. Moreover, $E(W|\mathbf{Z}_0^* = \mathbf{e}_l) = w_l$ for all $l = 1, 2, \cdots, K$.

As a consequence of Theorem 4.1 and Theorem 4.2, we obtain the spread rate for the random spread model.

Theorem 4.3. *Let $\rho = \rho(\mathbf{M}) > 1$ be the maximal eigenvalue of $\mathbf{M}$ and let $\mathbf{w}$ and $\mathbf{v}$ be the normalized left and right eigenvectors of $\mathbf{M}$ associated with ρ. If the matrix M is primitive and the covariance matrices $\mathbf{V}^{(l)}$'s are finite, then for every $l = 1, 2, \cdots, K-1$, on the event $\left\{\|\mathbf{Z}_n^*\| \big| \mathbf{Z}_0^* = \mathbf{e}_l \to \infty\right\}$, the spread rate*

$$s_{a_l}^f(a_j) = v_j > 0 \quad \textit{for } j = 1, 2, \cdots, K$$

with probability 1. If $l = K$, then the spread rate

$$s_{a_l}^f(a_j) = \begin{cases} 0, & \textit{for } j = 1, 2, \cdots, K-1; \\ 1, & \textit{for } j = K \end{cases}$$

with probability 1.

Remark 4.4. *The results in Theorem 4.1, Theorem 4.2 and Theorem 4.3 should be able to be extended to the cases when the decomposition of the offspring mean matrix has primitive or irreducible components as stated in the theorems in Chapter 3 by adopting the similar methods in the proofs in Section 4.2.*

In particular, we state the results in Theorem 4.5 for the case where $r = 2$, i.e., there are two primitive components in the matrix

$$\mathbf{M} = \begin{bmatrix} A^{[11]} & O & O \\ C & A^{[22]} & O \\ D & E & 1 \end{bmatrix}.$$

where the $d \times d$ matrix $A^{[11]}$ and the $(K-d-1)\times(K-d-1)$ matrix $A^{[22]}$ are primitive. Note that, if $\rho(A^{[22]}) > \rho(A^{[11]}) > 1$ then $v_i = 0$ for $i = 1, 2, \cdots, d$, $v_i > 0$ for $i = d+1, \cdots, K$, $w_i > 0$ for $i = 1, 2, \cdots, K-1$ and $w_K = 0$. On the other hand, if $\rho(A^{[11]}) > \rho(A^{[22]}) > 1$ then $v_i > 0$ for $i = 1, 2, \cdots, K$, $w_i > 0$ for $i = 1, \cdots, d$ and $w_i = 0$ for $i = d+1, \cdots, K$.

Theorem 4.5. *Let $\rho = \rho(\mathbf{M}) > 1$ be the maximal eigenvalue of $\mathbf{M}$ and let $\mathbf{w}$ and $\mathbf{v}$ be the normalized left and right eigenvectors of $\mathbf{M}$ associated with ρ. If the matrix M has two primitive components and is of the form (2) with $r = 2$ and the covariance matrices $\mathbf{V}^{(l)}$'s are finite for all $l = 1, 2, \cdots, K$, then*

(i) there exist a random variable W such that

$$\frac{\mathbf{Z}_n^*}{\rho^n} \to \mathbf{v}^t W \qquad \text{as } n \to \infty$$

both in mean square and with probability 1 and $E(W|\mathbf{Z}_0^* = \mathbf{e}_l) = w_l$ for all $l = 1, 2, \cdots, K$;

(ii) for any $l = 1, 2, \cdots, K-1$ and on the event $\Big\{\|\mathbf{Z}_n^*\| \big| \mathbf{Z}_0^* = \mathbf{e}_l \to \infty\Big\}$, if $\rho(A^{[22]}) > \rho(A^{[11]}) > 1$, then the spread rate

$$s_{a_l}^f(a_j) = v_j \quad \text{for } j = 1, 2, \cdots, K$$

with probability 1, and, if $\rho(A^{[11]}) > \rho(A^{[22]}) > 1$, then the spread rate

$$s_{a_l}^f(a_j) = \begin{cases} v_j, & \text{for } l = 1, 2, \cdots, d \text{ and } j = 1, 2, \cdots, K; \\ 0, & \text{for } l = d+1, \cdots, K-1 \text{ and } j = 1, 2, \cdots, d; \\ v'_{j-d}, & \text{for } l = d+1, \cdots, K-1 \text{ and } j = d+1, \cdots, K \end{cases}$$

with probability 1, where $\mathbf{v}' = (v'_i)$ is the normalized right eigenvector of the $(K-d)\times(K-d)$ matrix

$$M' = \begin{bmatrix} A^{[22]} & O \\ E & 1 \end{bmatrix}.$$

associated with the maximal eigenvalue $\rho(M')$ such that $\sum_{i=1}^{K-d} v'_i = 1$. For $l = K$, the spread rate

$$s_{a_l}^f(a_j) = \begin{cases} 0, & \text{for } j = 1, 2, \cdots, K-1; \\ 1, & \text{for } j = K \end{cases}$$

with probability 1

4.2 Proofs of main theorems for random spread models

In order to give the proofs for Theorem 4.1, Theorem 4.2 and Theorem 4.3 in Section 4.1, we need the following lemmas. Throughout Section 4.2, we assume that the hypotheses in Theorem 4.3 hold and let ρ, $\mathbf{w}$ and $\mathbf{v}$ be defined as in Theorem 4.2.

Since the offspring mean matrix describes the evolution of the branching population on average, we first investigate the growth rate of the mean matrix $\mathbf{M}^n$ in the following lemma:

Lemma 4.6. *For any $j = 1, 2, \cdots, K$, we have that*

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{e}_i}{\rho^n} = \begin{cases} w_i v_j, & \text{for } i \neq K; \\ 0, & \text{for } i = K. \end{cases}$$

Proof. See Page 4 in Ban et al.[4]. □

Now, for each $l = 1, 2, \cdots, K$, we define the matrix $\mathbf{C}_n^{(l)}$ in which the (i, j)-entry is

$$E(Z^*_{n,i} Z^*_{n,j} | \mathbf{Z}^*_0 = \mathbf{e}_l)$$

for all $i, j = 1, 2, \cdots, K$. Note that this matrix $\mathbf{C}_n^{(l)}$ is symmetric and the only nonzero entry in $\mathbf{C}_0^{(l)}$ is the (l, l)-entry. By Harris [12] (Page 37), $\mathbf{C}_n^{(l)}$ has an iterated formula:

$$\mathbf{C}_{n+1}^{(l)} = \mathbf{M}\mathbf{C}_n^{(l)}\mathbf{M}^t + \sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z^*_{n,r} | \mathbf{Z}^*_0 = \mathbf{e}_l),$$

for $n = 0, 1, 2, \cdots$, and it implies the following:

$$\mathbf{C}_n^{(l)} = \mathbf{M}^n \mathbf{C}_0^{(l)} (\mathbf{M}^t)^n + \sum_{s=1}^{n} \mathbf{M}^{n-s} \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z^*_{s-1,r} | \mathbf{Z}^*_0 = \mathbf{e}_l) \big) (\mathbf{M}^t)^{n-s}.$$

For each $l = 1, 2, \cdots, K$ and each $n = 1, 2, \cdots$, let $\mathbf{I}_{n,1}^{(l)} = \mathbf{M}^n \mathbf{C}_0^{(l)} (\mathbf{M}^t)^n$ and

$$\mathbf{I}_{n,2}^{(l)} = \sum_{s=1}^{n} \mathbf{M}^{n-s} \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z^*_{s-1,r} | \mathbf{Z}^*_0 = \mathbf{e}_l) \big) (\mathbf{M}^t)^{n-s}$$

then $\mathbf{C}_n^{(l)} = \mathbf{I}_{n,1}^{(l)} + \mathbf{I}_{n,2}^{(l)}$.

By the formula of $\mathbf{C}_n^{(l)}$ and the result in Lemma 4.6, we can show that the elements in $\mathbf{C}_n^{(l)}$ grow like ρ^{2n} in Lemmas 4.7-4.9. This gives the convergence of the normalized population vector in mean square and becomes the key to prove the convergence in Theorem 4.1 and the property in Theorem 4.2.

Lemma 4.7. *For each $i, j = 1, 2, \cdots, K$ and $l = 1, 2, \cdots, K-1$,*

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{I}_{n,1}^{(l)} \mathbf{e}_i}{\rho^{2n}} = w_l^2 v_i v_j.$$

Proof. By Lemma 4.6, since $l \neq K$, we have

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{e}_l}{\rho^n} = w_l v_j$$

and hence

$$\lim_{n\to\infty} \frac{\mathbf{e}_l^t (\mathbf{M}^t)^n \mathbf{e}_i}{\rho^n} = \lim_{n\to\infty} \left(\frac{\mathbf{e}_i^t \mathbf{M}^n \mathbf{e}_l}{\rho^n} \right)^t = w_l v_i.$$

So,

$$\begin{aligned}
& \lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{I}_{n,1}^{(l)} \mathbf{e}_i}{\rho^{2n}} \\
= & \lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{C}_0^{(l)} (\mathbf{M}^t)^n \mathbf{e}_i}{\rho^{2n}} \\
= & \lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{e}_l}{\rho^n} \cdot \mathbf{e}_l^t \mathbf{C}_0^{(l)} \mathbf{e}_l \cdot \frac{\mathbf{e}_l^t (\mathbf{M}^t)^n \mathbf{e}_i}{\rho^n} \\
= & \lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{e}_l}{\rho^n} \cdot \lim_{n\to\infty} \frac{\mathbf{e}_l^t (\mathbf{M}^t)^n \mathbf{e}_i}{\rho^n} \\
= & w_l v_j \cdot w_l v_i \\
= & w_l^2 v_i v_j.
\end{aligned}$$

□

Lemma 4.8. *If $l \neq K$, then*

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{I}_{n,2}^{(l)} \mathbf{e}_i}{\rho^{2n}} = w_l^2 v_i v_j A$$

for some non-negative constant A.

Proof. By Lemma 4.6, for any $\epsilon > 0$, there exists an $N = N(i, j, l, \epsilon) \in \mathbb{N}$ such that for every $n \geq N$, we have that

$$\left| \frac{\mathbf{e}_j^t \mathbf{M}^n \mathbf{e}_l}{\rho^n} \cdot \frac{\mathbf{e}_l^t (\mathbf{M}^t)^n \mathbf{e}_i}{\rho^n} - w_l^2 v_i v_j \right| < \epsilon.$$

For $n > 2N$,

$$\begin{aligned}
&\frac{\mathbf{e}_j^t \mathbf{I}_{n,2}^{(l)} \mathbf{e}_i}{\rho^{2n}} \\
= &\frac{1}{\rho^{2n}} \mathbf{e}_j^t \Big[\sum_{s=1}^{n} \mathbf{M}^{n-s} \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l) \big) (\mathbf{M}^t)^{n-s} \Big] \mathbf{e}_i \\
= &\sum_{s=1}^{N} \frac{1}{\rho^{2s}} \frac{\mathbf{e}_j^t \mathbf{M}^{n-s} \mathbf{e}_l}{\rho^{n-s}} \cdot \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l) \big) \mathbf{e}_l \cdot \frac{\mathbf{e}_l^t (\mathbf{M}^t)^{n-s} \mathbf{e}_i}{\rho^{n-s}} \\
&+ \sum_{s=N+1}^{n-N} \frac{1}{\rho^{s+1}} \frac{\mathbf{e}_j^t \mathbf{M}^{n-s} \mathbf{e}_l}{\rho^{n-s}} \cdot \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} \frac{E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l)}{\rho^{s-1}} \big) \mathbf{e}_l \cdot \frac{\mathbf{e}_l^t (\mathbf{M}^t)^{n-s} \mathbf{e}_i}{\rho^{n-s}} \\
&+ \frac{1}{\rho^n} \sum_{s=n-N+1}^{n} \frac{1}{\rho^{n-s+1}} \mathbf{e}_j^t \mathbf{M}^{n-s} \mathbf{e}_l \cdot \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} \frac{E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l)}{\rho^{s-1}} \big) \mathbf{e}_l \cdot \mathbf{e}_l^t (\mathbf{M}^t)^{n-s} \mathbf{e}_i \\
\equiv &J_{n,1}^{(l)} + J_{n,2}^{(l)} + J_{n,3}^{(l)}.
\end{aligned}$$

First of all, if $s \le N$, then $n - s > N$ and $s - 1 < N$. Let

$$A_1 \equiv \sum_{s=1}^{N} \frac{1}{\rho^{2s}} \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l) \big) \mathbf{e}_l$$

and we have that $0 \le A_1 < \infty$ and

$$J_{n,1}^{(l)} \to w_l^2 v_i v_j A_1, \qquad \text{as } n \to \infty.$$

Secondly, if $N+1 \le s \le n-N$ and $n > 2N$, then $n-s \ge N$ and $s-1 \ge N$. Hence, let

$$A_2 \equiv \sum_{s=N+1}^{\infty} \frac{1}{\rho^{s+1}} w_l \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} v_r \big) \mathbf{e}_l$$

and note that, since $\rho > 1$, A_2 is a non-negative and convergent series. Therefore, we have that

$$J_{n,2}^{(l)} \to w_l^2 v_i v_j A_2, \qquad \text{as } n \to \infty.$$

Thirdly, we notice that

$$\frac{E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l)}{\rho^{s-1}} = \frac{\mathbf{e}_r^t \mathbf{M}^{s-1} \mathbf{e}_l}{\rho^{s-1}}$$

which converges to $w_l v_r$ as $s \to \infty$. Moreover, if $n-N+1 \le s \le n$ and $n > 2N$, then $0 \le n-s \le N-1$ and $s-1 > N$. So, there exists a constant $A_3 > 0$ such that the sum

$$\sum_{s=n-N+1}^{n} \frac{1}{\rho^{n-s+1}} \mathbf{e}_j^t \mathbf{M}^{n-s} \mathbf{e}_l \cdot \mathbf{e}_l^t \big(\sum_{r=1}^{K} \mathbf{V}^{(r)} \frac{E(Z_{s-1,r}^* | \mathbf{Z}_0^* = \mathbf{e}_l)}{\rho^{s-1}} \big) \mathbf{e}_l \cdot \mathbf{e}_l^t (\mathbf{M}^t)^{n-s} \mathbf{e}_i$$

is non-negative and bounded by A_3 and hence $J_{n,3}^{(l)} = O(\rho^{-n})$ as $n \to \infty$.

Therefore, we have that

$$\frac{\mathbf{e}_j^t \mathbf{I}_{n,2}^{(l)} \mathbf{e}_i}{\rho^{2n}} = J_{n,1}^{(l)} + J_{n,2}^{(l)} + J_{n,3}^{(l)} \to w_l^2 v_i v_j A, \qquad \text{as } n \to \infty,$$

where $A = A_1 + A_2 \geq 0$. □

Lemma 4.7 and Lemma 4.8 together give the following: if $l \neq K$, then there exists a constant $A' = A + 1 > 0$ such that

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{C}_n^{(l)} \mathbf{e}_i}{\rho^{2n}} = w_l^2 v_i v_j A'$$

for all $i, j = 1, 2, \cdots, K$ and $l = 1, 2, \cdots, K-1$.

On the other hand, the limit when $l = K$ is straight forward from Lemma 4.6:

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{C}_n^{(K)} \mathbf{e}_i}{\rho^{2n}} = 0.$$

Notice that $w_K = 0$ and then the following lemma is concluded.

Lemma 4.9. *For all $i, j, l = 1, 2, \cdots, K$, there exists a constant $A' > 0$ such that*

$$\lim_{n\to\infty} \frac{\mathbf{e}_j^t \mathbf{C}_n^{(l)} \mathbf{e}_i}{\rho^{2n}} = w_l^2 v_i v_j A'$$

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. For $n = 0, 1, 2, \cdots$, let $\mathbf{W}_n = \dfrac{\mathbf{Z}_n^*}{\rho^n}$ and we have that

$$E(\mathbf{W}_n^t \mathbf{W}_n | \mathbf{Z}_0^* = \mathbf{e}_l) = \frac{E((\mathbf{Z}_n^*)^t \mathbf{Z}_n^* | \mathbf{Z}_0^* = \mathbf{e}_l)}{\rho^{2n}} = \frac{\mathbf{C}_n^{(l)}}{\rho^{2n}}.$$

Since $\dfrac{\mathbf{e}_j^t \mathbf{C}_n^{(l)} \mathbf{e}_i}{\rho^{2n}}$ converges as $n \to \infty$ for all $i, j, l = 1, 2, \cdots, K$, every element in $E(\mathbf{W}_n^t \mathbf{W}_n | \mathbf{Z}_0^* = \mathbf{e}_l)$ is finite for all $n = 0, 1, 2, \cdots$ and all $l = 1, 2, \cdots, K$.

We will first show that

$$E\big((\mathbf{W}_{n+m} - \mathbf{W}_n)^t (\mathbf{W}_{n+m} - \mathbf{W}_n)\big| \mathbf{Z}_0^* = \mathbf{e}_l\big) \to 0, \qquad \text{as } m, n \to \infty.$$

Note that

$$\begin{aligned}
& E\big((\mathbf{W}_{n+m} - \mathbf{W}_n)^t (\mathbf{W}_{n+m} - \mathbf{W}_n)\big| \mathbf{Z}_0^* = \mathbf{e}_l\big) \\
= \; & E\big(\mathbf{W}_{n+m}^t \mathbf{W}_{n+m} - \mathbf{W}_{n+m}^t \mathbf{W}_n - \mathbf{W}_n^t \mathbf{W}_{n+m} + \mathbf{W}_n^t \mathbf{W}_n \big| \mathbf{Z}_0^* = \mathbf{e}_l\big) \\
= \; & E\big(\mathbf{W}_{n+m}^t \mathbf{W}_{n+m} \big| \mathbf{Z}_0^* = \mathbf{e}_l\big) - E\big(\mathbf{W}_{n+m}^t \mathbf{W}_n \big| \mathbf{Z}_0^* = \mathbf{e}_l\big) \\
& - E\big(\mathbf{W}_n^t \mathbf{W}_{n+m} \big| \mathbf{Z}_0^* = \mathbf{e}_l\big) + E\big(\mathbf{W}_n^t \mathbf{W}_n \big| \mathbf{Z}_0^* = \mathbf{e}_l\big)
\end{aligned}$$

and that, from the Markov property of branching process, $E(\mathbf{Z}^*_{n+m}|\mathbf{Z}^*_n) = \mathbf{Z}^*_n(\mathbf{M}^t)^m$ for $n, m = 0, 1, 2, \cdots$. So, we have that

$$\begin{aligned}
& E\big(\mathbf{W}^t_{n+m}\mathbf{W}_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big((\mathbf{Z}^*_{n+m})^t\mathbf{Z}^*_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big(E((\mathbf{Z}^*_{n+m})^t\mathbf{Z}^*_n|\mathbf{Z}^*_n)\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big(E((\mathbf{Z}^*_{n+m})^t|\mathbf{Z}^*_n)\mathbf{Z}^*_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big(\mathbf{M}^m(\mathbf{Z}^*_n)^t\mathbf{Z}^*_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{\mathbf{M}^m}{\rho^{2n+m}} E\big((\mathbf{Z}^*_n)^t\mathbf{Z}^*_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{\mathbf{M}^m\mathbf{C}^{(l)}_n}{\rho^{2n+m}}
\end{aligned}$$

and

$$\begin{aligned}
& E\big(\mathbf{W}^t_n\mathbf{W}_{n+m}\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big((\mathbf{Z}^*_n)^t\mathbf{Z}^*_{n+m}\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big(E((\mathbf{Z}^*_n)^t\mathbf{Z}^*_{n+m}|\mathbf{Z}^*_n)\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big((\mathbf{Z}^*_n)^tE(\mathbf{Z}^*_{n+m}|\mathbf{Z}^*_n)\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big((\mathbf{Z}^*_n)^t\mathbf{Z}^*_n(\mathbf{M}^t)^m\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big) \\
= \; & \frac{1}{\rho^{2n+m}} E\big((\mathbf{Z}^*_n)^t\mathbf{Z}^*_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big)(\mathbf{M}^t)^m \\
= \; & \frac{\mathbf{C}^{(l)}_n(\mathbf{M}^t)^m}{\rho^{2n+m}}.
\end{aligned}$$

By the same arguments as in the proofs of Lemma 4.7 and Lemma 4.8, we can show that, as $m, n \to \infty$,

$$\begin{aligned}
& \frac{\mathbf{e}^t_j\mathbf{M}^m\mathbf{C}^{(l)}_n\mathbf{e}_i}{\rho^{2n+m}} \\
= \; & \frac{\mathbf{e}^t_j\mathbf{M}^{n+m}\mathbf{e}_l}{\rho^{n+m}} \cdot \mathbf{e}^t_l\mathbf{C}^{(l)}_0\mathbf{e}_l \cdot \frac{\mathbf{e}^t_l(\mathbf{M}^t)^n\mathbf{e}_i}{\rho^n} \\
& + \frac{1}{\rho^{2n+m}}\mathbf{e}^t_j\Big[\sum_{s=1}^{n}\mathbf{M}^{n+m-s}\big(\sum_{r=1}^{K}\mathbf{V}^{(r)}E(Z^*_{s-1,r}|\mathbf{Z}^*_0 = \mathbf{e}_l)\big)(\mathbf{M}^t)^{n-s}\Big]\mathbf{e}_i \\
\to \; & w^2_l v_i v_j A'
\end{aligned}$$

and, similarly, $\dfrac{\mathbf{e}^t_j\mathbf{C}^{(l)}_n(\mathbf{M}^t)^m\mathbf{e}_i}{\rho^{2n+m}} \to w^2_l v_i v_j A'$ where the constant A' is as defined in Lemma 4.9. So, as $m, n \to \infty$,

$$\mathbf{e}^t_j E\big(\mathbf{W}^t_{n+m}\mathbf{W}_n\big|\mathbf{Z}^*_0 = \mathbf{e}_l\big)\mathbf{e}_i \to w^2_l v_i v_j A'$$

and

$$\mathbf{e}_j^t E\big(\mathbf{W}_n^t \mathbf{W}_{n+m} \big| \mathbf{Z}_0^* = \mathbf{e}_l\big)\mathbf{e}_i \to w_l^2 v_i v_j A'$$

and thus, as $m, n \to \infty$,

$$\begin{aligned} & \mathbf{e}_j^t E\big((\mathbf{W}_{n+m} - \mathbf{W}_n)^t (\mathbf{W}_{n+m} - \mathbf{W}_n) \big| \mathbf{Z}_0^* = \mathbf{e}_l\big)\mathbf{e}_i \\ = & \frac{\mathbf{e}_j^t \mathbf{C}_{n+m}^{(l)} \mathbf{e}_i}{\rho^{2(n+m)}} - \frac{\mathbf{e}_j^t \mathbf{M}^m \mathbf{C}_n^{(l)} \mathbf{e}_i}{\rho^{2n+m}} - \frac{\mathbf{e}_j^t \mathbf{C}_n^{(l)} (\mathbf{M}^t)^m \mathbf{e}_i}{\rho^{2n+m}} + \frac{\mathbf{e}_j^t \mathbf{C}_n^{(l)} \mathbf{e}_i}{\rho^{2n}} \to 0. \end{aligned}$$

Therefore, the sequence $\{\mathbf{W}_n\}_{n\geq 0}$ is a Cauchy and hence convergent in mean square. So, there exists a random vector $\mathbf{W}$ with $E(\mathbf{W}^t\mathbf{W})$ is finite such that

$$\mathbf{W}_n = \frac{\mathbf{Z}_n^*}{\rho^n} \to \mathbf{W} \text{ in mean square, as } n \to \infty.$$

Moreover, as $n \to \infty$,

$$\begin{aligned} & \mathbf{e}_j^t E\big((\mathbf{W} - \mathbf{W}_n)^t (\mathbf{W} - \mathbf{W}_n) \big| \mathbf{Z}_0^* = \mathbf{e}_l\big)\mathbf{e}_i \\ = & \lim_{m\to\infty} \mathbf{e}_j^t E\big((\mathbf{W}_m - \mathbf{W}_n)^t (\mathbf{W}_m - \mathbf{W}_n) \big| \mathbf{Z}_0^* = \mathbf{e}_l\big)\mathbf{e}_i \\ = & O(\rho^{-2n}) \end{aligned}$$

and hence,

$$\begin{aligned} & E\Big(\mathbf{e}_j^t \big(\sum_{n=0}^{\infty} (\mathbf{W} - \mathbf{W}_n)^t (\mathbf{W} - \mathbf{W}_n)\big)\mathbf{e}_i \Big| \mathbf{Z}_0^* = \mathbf{e}_l\Big) \\ = & \sum_{n=0}^{\infty} E\Big(\mathbf{e}_j^t \big((\mathbf{W} - \mathbf{W}_n)^t (\mathbf{W} - \mathbf{W}_n)\big)\mathbf{e}_i \Big| \mathbf{Z}_0^* = \mathbf{e}_l\Big) < \infty. \end{aligned}$$

So, given $\mathbf{Z}_0^* = \mathbf{e}_l$, for all l, $\mathbf{e}_j^t \big(\sum_{n=0}^{\infty} (\mathbf{W} - \mathbf{W}_n)^t (\mathbf{W} - \mathbf{W}_n)\big)\mathbf{e}_i < \infty$ with probability 1 and, therefore, for all $i, j = 1, 2, \cdots, K$,

$$\lim_{n\to\infty} \mathbf{e}_j^t (\mathbf{W} - \mathbf{W}_n)^t (\mathbf{W} - \mathbf{W}_n)\mathbf{e}_i = 0 \qquad \text{with probability 1.}$$

It follows that $\mathbf{e}_j^t(\mathbf{W} - \mathbf{W}_n)\mathbf{e}_i \to 0$ with probability 1 for all $i, j = 1, 2, \cdots, K$ and hence gives, as $n \to \infty$,

$$\mathbf{W}_n \to \mathbf{W} \qquad \text{with probability 1.}$$

So, $\mathbf{W}_n$ converges to $\mathbf{W}$ both in mean square and with probability 1 and the proof of Theorem 4.1 is complete. □

Next, we are going to prove Theorem 4.2 which says that the vectors $\mathbf{W}$ and $\mathbf{v}$ share the same direction.

Proof of Theorem 4.2. Let $\mathbf{W} = (W_1, W_2, \cdots, W_K)$, where $W_1, \cdots, W_K$ are random variables. Since

$$E(\mathbf{W}^t\mathbf{W}|\mathbf{Z}_0^* = \mathbf{e}_l) = \lim_{n\to\infty} E\Big(\frac{(\mathbf{Z}_n^*)^t}{\rho^n} \cdot \frac{\mathbf{Z}_n^*}{\rho^n}\Big|\mathbf{Z}_0^* = \mathbf{e}_l\Big) = \lim_{n\to\infty} \frac{\mathbf{C}_n^{(l)}}{\rho^{2n}},$$

by Lemma 4.9, we have that

$$\mathbf{e}_j^t E(\mathbf{W}^t\mathbf{W}|\mathbf{Z}_0^* = \mathbf{e}_l)\mathbf{e}_i = w_l^2 v_i v_j A'$$

for all $i, j = 1, 2, \cdots, K$ and hence

$$\begin{aligned}
& E\big((v_iW_j - v_jW_i)^2\big|\mathbf{Z}_0^* = \mathbf{e}_l\big) \\
= \;& v_i^2 E(W_j^2|\mathbf{Z}_0^* = \mathbf{e}_l) - 2v_iv_jE(W_iW_j|\mathbf{Z}_0^* = \mathbf{e}_l) + v_j^2E(W_i^2|\mathbf{Z}_0^* = \mathbf{e}_l) \\
= \;& v_i^2\mathbf{e}_j^tE(\mathbf{W}^t\mathbf{W}|\mathbf{Z}_0^* = \mathbf{e}_l)\mathbf{e}_j - 2v_iv_j\mathbf{e}_i^tE(\mathbf{W}^t\mathbf{W}|\mathbf{Z}_0^* = \mathbf{e}_l)\mathbf{e}_j \\
& +v_j^2\mathbf{e}_i^tE(\mathbf{W}^t\mathbf{W}|\mathbf{Z}_0^* = \mathbf{e}_l)\mathbf{e}_i \\
= \;& v_i^2w_l^2v_jv_jA' - 2v_iv_jw_l^2v_iv_jA' + v_j^2w_l^2v_iv_iA' \\
= \;& 0.
\end{aligned}$$

So, $v_iW_j = v_jW_i$ with probability 1, for all $i, j = 1, 2, \cdots, K$. Moreover, since all v_i's are positive, we have that

$$\frac{W_i}{v_i} = \frac{W_j}{v_j} \qquad \text{with probability 1}$$

for all $i, j = 1, 2, \cdots, K$. Therefore, there exists a random variable W such that

$$\mathbf{W} = (W_1, W_2, \cdots, W_k) = (v_1W, v_2W, \cdots, v_KW) = \mathbf{v}W$$

and

$$\frac{\mathbf{Z}_n^*}{\rho^n} \to \mathbf{v}W$$

with probability 1. Moreover, since

$$E(\mathbf{Z}_n^*|\mathbf{Z}_0^* = \mathbf{e}_l) = \mathbf{e}_l^t(\mathbf{M}^t)^n,$$

we have that

$$E(\mathbf{W}_n|\mathbf{Z}_0^* = \mathbf{e}_l) = \frac{1}{\rho^n}E(\mathbf{Z}_n^*|\mathbf{Z}_0^* = \mathbf{e}_l) = \frac{\mathbf{e}_l^t(\mathbf{M}^t)^n}{\rho^n}$$

and then

$$E(W_i|\mathbf{Z}_0^* = \mathbf{e}_l) = \lim_{n\to\infty} E(\mathbf{W}_n\mathbf{e}_i|\mathbf{Z}_0^* = \mathbf{e}_l) = \lim_{n\to\infty} \frac{\mathbf{e}_l^t(\mathbf{M}^t)^n\mathbf{e}_i}{\rho^n} = w_lv_i,$$

that is, $E(v_i W|\mathbf{Z}_0^* = \mathbf{e}_l) = w_l v_i$ and, therefore,

$$E(W|\mathbf{Z}_0^* = \mathbf{e}_l) = w_l$$

for all $l = 1, 2, \cdots, K$ and it completes the proof of Theorem 4.2.

□

Finally, the proof of Theorem 4.3 is straightforward from the results in Theorem 4.2:

Proof of Theorem 4.3. Given $\mathbf{Z}_0^* = \mathbf{e}_l$ with $l = 1, 2, \cdots, K-1$, we have that, for all $j = 1, 2, \cdots, K$,

$$\begin{aligned} s^f_{a_l}(a_j) &= \lim_{n\to\infty} \frac{Z^*_{n,j}}{\sum_{i=1}^K Z^*_{n,i}} = \lim_{n\to\infty} \frac{Z^*_{n,j}/\rho^n}{\sum_{i=1}^K Z^*_{n,i}/\rho^n} = \frac{W_j}{\sum_{i=1}^K W_i} \\ &= \frac{v_j W}{\sum_{i=1}^K v_i W} = \frac{v_j}{\sum_{i=1}^K v_i} = v_j \qquad \text{with probability 1.} \end{aligned}$$

If $l = K$, that is, $\mathbf{Z}_0^* = \mathbf{e}_K$, then this model starts with an individual of frozen type a_K. So, in this case, $\mathbf{Z}_n^* = \mathbf{e}_K$ with probability 1 for all $n = 0, 1, 2, \cdots$. Therefore,

$$s^f_{a_K}(a_j) = \lim_{n\to\infty} \frac{Z^*_{n,j}}{\sum_{i=1}^K Z^*_{n,i}} = \frac{Z^*_{n,j}}{Z^*_{n,K}} = \begin{cases} 0, & \text{if } j = 1, 2, \cdots, K-1 \\ 1, & \text{if } j = K. \end{cases}$$

with probability 1 and the proof is complete.

□

5 Numerical results

The main purpose of the section is to demonstrate the algorithm provided in Section 3 as well as the numerical simulations for the random spread models.

5.1 The topological case

Let matrix $M = [A^{[ij]}]_{1 \le i,j \le 5}$ be defined with

$$A^{[11]} = A^{[44]} = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, A^{[22]} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix},$$

$$A^{[33]} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, A^{[55]} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$A^{[31]} = A^{[32]} = A^{[41]} = A^{[43]} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A^{[53]} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and all other $A^{[ij]}$'s are zero matrices. We note that each $A^{[ii]}$ is a primitive matrix with period two, and the characteristic matrix χ_M is

$$\chi_M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

with the graph representation provided in Figure 5, where each node is labeled by its corresponding block matrix $A^{[ii]}$ and eigenvalues ρ_j ($\rho_1 = \frac{1+\sqrt{5}}{2} > \sqrt{2} = \rho_2$). For the purpose of illustration, we analyze in the following subsections the asymptotic behavior of the population when there is a unique ancestor $b \in V^{[11]} \cup V^{[22]}$.

5.1.1 Size of the population

Note that the only maximal paths initiating from $V^{[11]}$ in the tree structure are

$$V^{[11]}V^{[44]}, V^{[11]}V^{[33]}V^{[33]}, V^{[11]}V^{[33]}V^{[55]},$$

and the maximal number of appearances of maximal eigenvalues in each path is 1. Since $A^{[ii]}$'s have period 2, according to Lemma 3.4 and Theorem 3.2, we know the size of the population is $O(\rho_1^n)$ asymptotically, where O denotes the big O notation. Similarly, if one applies the same argument to the case $b \in V^{[22]}$, the maximal number of appearances of maximal eigenvalues in each path is 2, and the size of the population is $O(n\rho_1^{n-1})$. The simulation is provided as in Figure 6, where the dashed line denotes the asymptotic value of the population.

5.1.2 Spread rate

Firstly, we consider the case when $b \in V^{[11]}$. Based on Theorem 3.2, since the period of $A^{[ii]}$ is 2, for each $1 \leq i \leq 2$ there is a set P_i such that $S_b(a) > 0$

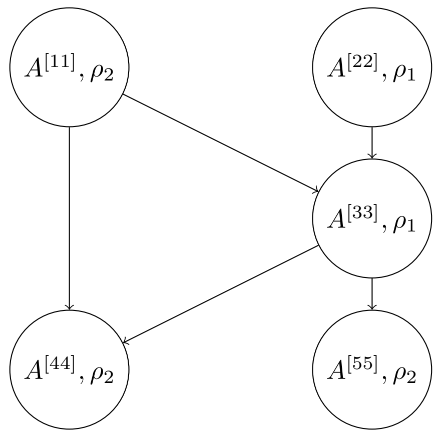
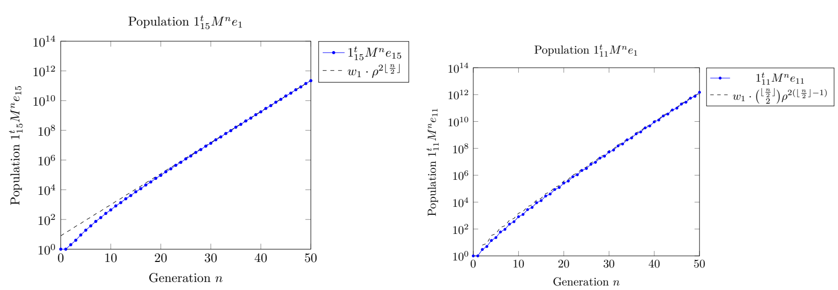


Figure 5: tree structure of block matrix M



(a) $b \in V^{[11]}$

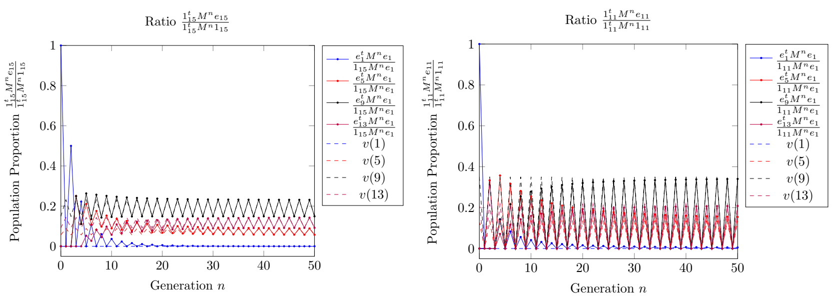
(b) $b \in V^{[22]}$

Figure 6: size of the population

for all $a \in P_i(a)$. In fact, it can be verified by looking at the eigenvectors of $P_1 \cup P_2 = V^{[33]} \cup V^{[44]} \cup V^{[55]}$. Similarly, when $b \in V^{[22]}$, we also have $P_1 \cup P_2 = V^{[33]} \cup V^{[44]} \cup V^{[55]}$. The simulation is provided as in Figure 7, where the periodic fluctuation is clearly observable in the figure.

5.2 Random models

To demonstrated Theorems 4.1- 4.5, we provide three numerical examples.



(a) $b \in V^{[11]}$ (b) $b \in V^{[22]}$

Figure 7: spread rate

5.2.1 The case with only one primitive component

we consider the following random spread model with probability distribution vectors $\{p^{(i)}(\cdot)\}_{i=1}^3$:

$$\begin{cases} p^{(1)}(3,1,1) = \frac{1}{2}, p^{(1)}(1,3,1) = \frac{1}{2}, \\ p^{(2)}(1,2,2) = \frac{1}{2}, p^{(2)}(3,2,0) = \frac{1}{2}, \\ p^{(3)}(1,0,0) = 1, \end{cases}$$

for which the associated offspring mean matrix can be computed as

$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix}. \tag{9}$$

The normalized right and left eigenvectors of $\mathbf{M}$ associated with the maximal eigenvalue $\rho = 4$ are $\mathbf{v} = (3/8, 3/8, 1/4)$ and $\mathbf{w} = (4/3, 4/3, 0)$, respectively,. It then can be inferred from Theorem 4.2 that $E(\mathbf{Z}_n^*/\rho^n|\mathbf{Z}_0^* = \mathbf{e}_1) \to w_1\mathbf{v}$. This can be observed in Figure 8, in which $E(\mathbf{Z}_n^*/\rho^n|\mathbf{Z}_0^* = \mathbf{e}_1) \to w_1\mathbf{v}$ is numerically attained by the empirical average over 50 realizations. The results shown in the figure are consistent with Theorem 4.3 in the sense that the asymptotic value of $E(\mathbf{Z}_n^*/\rho^n|\{\mathbf{Z}_0^* = \mathbf{e}_1\})$ is a positive vector. On the other hand, since $\mathbf{v}$ and w_1 are positive, Theorem 4.1 and Theorem 4.2 implies the population size, on average, should be $w_1\rho^n$ at generation n.

5.2.2 The case with two primitive components

Similar discussions extend to more complex random models and the following two numerical examples also show the consistency with the results in Theorem

4.5. Consider the spread model $\{p^{(i)}(\cdot)\}_{i=1}^{5}$.

$$\begin{cases} p^{(1)}(2,1,0,3,0) = \frac{1}{2}, p^{(1)}(0,1,2,3,2) = \frac{1}{2} \\ p^{(2)}(3,3,2,1,1) = \frac{1}{3}, p^{(2)}(0,0,2,1,1) = \frac{2}{3} \\ p^{(3)}(0,0,3,2,0) = \frac{1}{2}, p^{(3)}(0,0,1,2,2) = \frac{1}{2} \\ p^{(4)}(0,0,4,1,4) = \frac{1}{3}, p^{(4)}(0,0,2,1,3) = \frac{1}{3}, p^{(4)}(0,0,0,4,2) = \frac{1}{3} \\ p^{(5)}(0,0,0,0,1) = 1 \end{cases}$$

for which the associated offspring mean matrix can be computed as

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 2 & 0 \\ 3 & 1 & 2 & 2 & 0 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix} \tag{10}$$

Similar results as above are illustrated in Figure 9. It is noteworthy that under this setting, the ratios $\mathbf{Z}_1^*/\rho^n$ and $\mathbf{Z}_1^*/\rho^n$ tend to zero, since the submatrix indexed by the symbols (namely, $A^{[11]} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$) corresponds to a primitive component with its maximal eigenvalue strictly smaller than that of $\mathbf{M}$. This should be compared with the following random spread model.

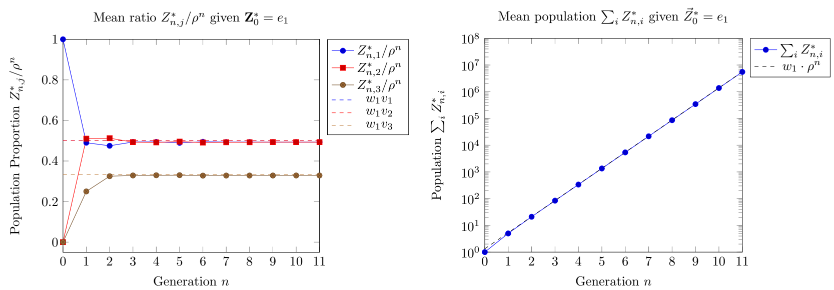
Consider the spread model $\{p^{(i)}(\cdot)\}_{i=1}^{5}$.

$$\begin{cases} p^{(1)}(4,1,1,1,0) = \frac{1}{4}, p^{(1)}(0,1,3,3,2) = \frac{1}{4}, p^{(1)}(2,3,0,4,1) = \frac{1}{2} \\ p^{(2)}(1,3,2,0,1) = \frac{2}{3}, p^{(2)}(4,0,2,3,1) = \frac{1}{3} \\ p^{(3)}(0,0,2,0,1) = \frac{1}{2}, p^{(3)}(0,0,0,1,2) = \frac{1}{2}, p^{(3)}(0,0,1,2,0) = \frac{1}{2} \\ p^{(4)}(0,0,0,1,4) = \frac{2}{3}, p^{(4)}(0,0,3,1,1) = \frac{1}{3} \\ p^{(5)}(0,0,0,0,1) = 1 \end{cases}$$

for which the associated offspring mean matrix is can be computed as

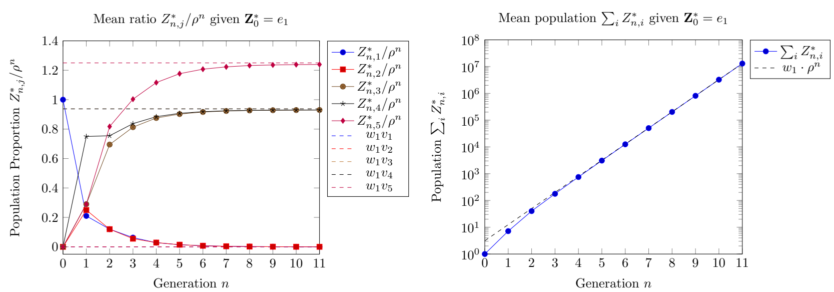
$$\mathbf{M} = \begin{bmatrix} 2 & 2 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 3 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix} \tag{11}$$

It is to be noted that the only difference in the offspring mean matrices (10) and (11) lies in the submatrices $A^{[11]} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ and $A^{[22]} = \begin{bmatrix} m_{33} & m_{34} \\ m_{43} & m_{44} \end{bmatrix}$. This difference then leads to the positiveness of the asymptotic value $\mathbf{Z}_n^*/\rho^n$ in Figure 10, as opposed to Figure 9.



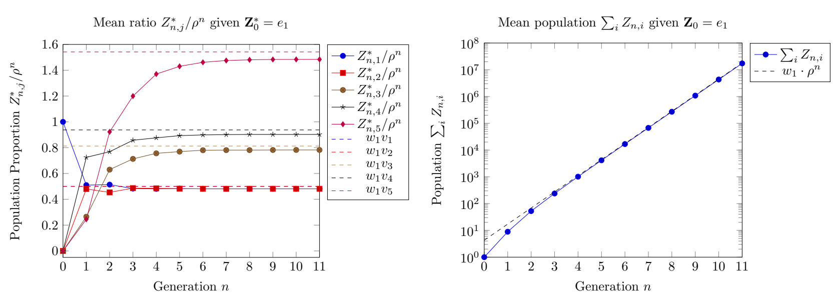
(a) $\mathbf{Z}^*_n/\rho^n$ (b) $\sum_i Z^*_{n,i}$

Figure 8: Example of a random model with three types. The mean ratio and the mean population are numerically approximated by empirical averages over 50 realizations.



(a) $\mathbf{Z}^*_n/\rho^n$ (b) $\sum_i Z^*_{n,i}$

Figure 9: Example of a random model with five types. The mean ratio and the mean population are numerically approximated by empirical averages over 50 realizations.



(a) $\mathbf{Z}^*_n/\rho^n$ (b) $\sum_i Z^*_{n,i}$

Figure 10: Example of a random model with five types. The mean ratio and the mean population are numerically approximated by empirical averages over 50 realizations.

6 Conclusion

Pandemic occurrences cause a lot of medical, economic and social problems. In order to reduce their impact, many prevention and control measures are applied to slow the spread of disease. For example, quarantine is a measure to keep people who have been in close contact with the infected apart from others, to avoid onward transmission, while isolation refers to the separation of confirmed or suspected cases from others for the duration of the infectious period. When such kind of measures are adopted, the chances of people with higher contagious ability spreading the virus may be limited. Therefore, the viral spread pattern will be changed, leading to different results in terms of the spread rate.

To model this phenomenon, we propose two mathematical models from the topological and random perspectives by means of substitution dynamical systems and the theory of branching processes. In both models, a so-called frozen symbol (type) is used to represent the group of individuals who have been isolated. We assume that this frozen symbol can only produce exactly one symbol of the same kind (or one offspring of the same type), namely, once a symbol is frozen, it has no ability to spread. In this work, we investigate the spread rate in both topological and random spread models with frozen symbols. In Chapter 3, we prove, for the topological spread model, the limiting spread rate exists and construct an algorithm to determine when the spread rate is positive for the case when the ξ-matrix can be decomposed into primitive components (Theorem 3.1, Proposition 3.5) and the case when the components are irreducible (Theorem 3.2, Proposition 3.7). In Chapter 4, we extend the classic results for branching processes with primitive offspring mean matrices to the random model with a frozen symbol. We prove the geometric growth of the population size (Theorem 4.1), show that the limit of type composition is proportional to the right eigenvector associated with the maximal eigenvalue of the offspring mean matrix (Theorem 4.2) and we find the spread rates (Theorem 4.3). In Chapter 5, some topological and random examples are provided. The numerical results support the theoretical results in Chapter 3 and Chapter 4.

The significance of our work is as follows: First of all, two spread models proposed here are discrete, so all the spread patterns of different types can be clearly represented by a deterministic matrix and hence the transmission or the spread at any time n is computable. In particular, this characteristic allows us to divide the types into group according to the accessibility to each other. Hence, the trace of the possible evolution from type to type can be determined and it leads to the proposed algorithm in this work for identifying the types with positive spread rates. Secondly, due to the structure of the models, the transition after time n can be represented as a power $\mathbf{M}^n$ of the initial ξ-matrix or offspring mean matrix, so the long-term behavior of the type composition is predictable using the information from $\mathbf{M}$. Namely, the right eigenvector associated with the spectral eigenvalue will give us the limiting spread rate of theses models. This is an advantage which provides us an easier way to find the spread rate without involving in any complicated computation or iterations. Thirdly, these models based on the substitution and branching processes can be alternated and generalized easily to fit different spread mechanisms. For example, the m-spread models [4] describes the dependence of the spread patterns and the spread models with frozen symbols in this work and [5] deals with the situations when some types are blocked. Finally, it is worth mentioning that, in this work, we derive a method to deal with the non-primitive ξ-matrix and offspring mean matrix. In the classic theories (both in substitution and branching processes), the primitive property is the key sufficient condition to study the limit behavior of the matrix using its maximal eigenvalue and the corresponding eigenvectors. However, our model setting with frozen symbols leads to the non-primitive cases. Therefore, the difficulty arises during the computation. In dealing with it, different cases are considered such as the matrices whose components are primitive or irreducible with some periods. In each case, we investigate the properties of its components to study the limit behavior of the whole matrix as well as deriving the algorithms to locate the positive spread rates. Further, these topological results provides the ideas to study the random spread models.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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