MATHEMATICAL ANALYSIS OF TOPOLOGICAL AND RANDOM m-ORDER SPREAD MODELS

JUNG-CHAO BAN, JYY-I HONG, AND YU-LIANG WU

ABSTRACT. This paper focuses on the analysis of two particular models, from deterministic and random perspective respectively, for spreading processes. With a proper encoding of propagation patterns, the spread rate of each pattern is discussed for both models by virtue of the substitution dynamical systems and branching process. In view of this, we are empowered to draw a comparison between two spreading processes according to their spreading models, based on which explanations are proposed on a higher frequency of a pattern in one model than the other. These results are then supported by the numerical evidence later in the article.

CONTENTS

1. INTRODUCTION

Occasionally, a pandemic spreads rapidly and widely around the world. To minimize the impact caused by these diseases, developing a mathematical model

Key words and phrases. topological spread model, random spread model, transition model, spread rate, n-spread rate.

This work was supported by the Ministry of Science and Technology, ROC [grant numbers MOST 109-2115-M-004-002-MY2, MOST 109-2115-M-004 -001 -MY2].

based on real data as a criterion for delivering required verdicts is an essential starting point. There are many works discussing propagation models and their dynamics (cf. Billings et al (2002); Halle et al (1993); Zheng and Bonasera (2020); Merkin et al (2005); Jin and Zhao (2009); Al-Jararha and Ou (2013); Ou and Wu (2006); Gourley and Lou (2014); Alexander et al (2004); Wang and Zhao (2012); Ruan and Wu (2009)) and it is worth noting that most modeling is done by using the (partial) differential equations. Ban et al (2021) attempted to take a different approach to the development of spread models in which the individuals are classified into various categories (individuals in the same category are said to be of the same type) and the spread pattern of each type is fully described in both deterministic and random senses and only depends on its own type. Namely, two distinct perspectives were provided: the topological spread models with the help of the substitution dynamical system (cf. Queffélec (2010)) which is a branch of dynamical systems and the random spread models by the means of the branching process which is a branch of the probability (cf. Athreya and Ney (2004)). Ban et al. examined the long term behavior of the models and showed that the spread rates are predictable and related to the maximal eigenvalue and its corresponding eigenvector of the ξ-matrix and the offspring mean matrix in topological and random spread models, respectively. However, the setting of these spread models can not illustrate the situations with more complicated spreading phenomena. For example, in the most epidemics, individuals may have multiple contacts with many sources of infection and, in general, a contagious period usually lasts for a while until the fully recovery. Therefore, it is more realistic to consider that the spread patterns are affected by the types of the surrounding individuals during their contagious periods. In this paper, we aim to develop such spread models which can exhibit this spreading characteristic. More precisely, we propose both the topological and random models in which the spread patterns are affected by the type composition of the individuals within m "generations" (or m units of time), where m can be thought as the length of the contagious period, so they will be called the m-spread model throughout this paper and therefore the spread models introduced in Ban et al (2021) become the special cases with $m = 1$ and will be called the 1-spread models.

The major goals of our work in this paper are stated below:

1. For both models, we discuss the long-term behavior of a particular individual, namely, the 'spread rate'. Additionally, we would like to find the rigorous formula for the spread rate of an individual if possible.
2. For both models, we aim to identify the factors that cause a high spread rate, and show that removing the said factor would reduce the spread rate.

Fortunately, the aforementioned problems can be completely solved for both models. The work done for the 1-spread models in Ban et al (2021) provides a foundation for us to investigate the m-spread model. We will transform the topological and random m-spread models into their corresponding topological and random 1-spread models and find the spread rate of each type in the m-spread models by investigating the spread rates for the induced 1-spread models. Moreover, the relation between the topological and random m-spread models is also addressed. Although this research work was originally conceived to construct a model for the spread of some virus or disease in epidemiology, the m-spread model can also be applied in many different areas such as population genetics, ecology, etc. The

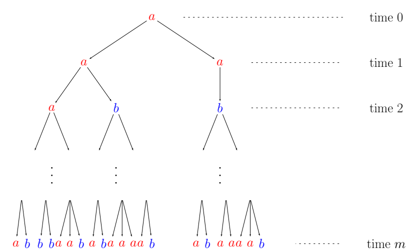


FIGURE 1. Illustration of an m-pattern

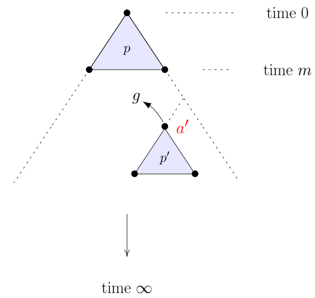


FIGURE 2. Illustration of an infinite spread model generated from a topological spread model

following two subsections provide a general description of both models and their main results.

1.1. **Topological m-spread models.** In this section, we provide a general description of the topological spread model, while detailed definitions and outcomes can be found in Section 2.

Let $\mathcal{A} = \{a, b\}$ be an alphabet[1], F be a finite, directed graph with root ϵ, and $p : F \to \mathcal{A}$ be an m-pattern (see Section 2) as Figure 1. Here and subsequently, p describes a certain manner of propagation described as follows. First, there is a type a individual at time-0, and it produces two a's at time-1. For the a on the left, it will produce one a individual and one b individual at time-2, while the a on the right will produce only one b at time-2. The same reasoning applies to time-n to time-$(n+1)$ for $0 \le n \le m-1$. Therefore, p demonstrates a certain mode of propagation of ancestor a from time-0 to time-m. Since $m \in \mathbb{N}$ is fixed and $\mathcal{S} = \{p_i\}_{i=1}^l$ is a collection of such m-patterns, i.e., a collection of spread modes, a *topological spread model* is a model such that each $p \in \mathcal{S}$ can be extended to an infinite spread mode with respect to $\mathcal{S}$, say τ_p, that is defined in an infinite (non-uniform) tree $T = T_p$ so that the restriction of τ_p to the first m-layers is p (see Figure 2). In addition, for each $g \in T_p$ associated with a type a' individual, the descendants of this individual from time-$(|g| + i)$ to time-$(|g| + i + 1)$ for $0 \le i \le m - 1$ is a spread mode of some $p' \in \mathcal{S}$, where $|g|$ stands for the length of g, i.e., the number of edges from ϵ to g.

For $p \in \mathcal{S}$ with type a individual at the root, the number $m \in \mathbb{N}$ indicates the 'length' of the spread mode from a, and the infinite spread mode τ_p specifies how p propagates according to $\mathcal{S}$ when time tends to infinity. Given $\mathcal{S}$, $p \in \mathcal{S}$ and $a \in \mathcal{A}$, the occurrence of a at τ_p from sth layer to tth layers for $s \le t$, is extremely complicated and random. For example, if a is a severe illness spreading in a pandemic, investigating the spread rate of a is one of the most important objectives of the spread model. Once the spread rate of some $a \in \mathcal{A}$ is calculated, we look for the primary spread modes (or factors) in $\mathcal{S}$ that cause the (high) spread rate of a and eliminate them to reduce the spread rate of a. Those are the two primary aims of this study, as previously mentioned. In Theorem 4, we propose

[1]We will call $\mathcal{A}$ the *type set* in Section 2

a methodology for calculating the spread rate of a symbol $a \in \mathcal{A}$. Furthermore, Theorem 5 provides a method to compute the spread rate of a pattern. Finally, Theorem 8 gives a comparison of the spread rates of two models, and such a result helps us to examine and identify the major underlying factor in the spread model that causes the (high) spread rate.

1.2. **Random m-spread models.** There is always uncertainty during propagation, and therefore it is reasonable to consider the case in which a mode may have a chance to produce different type compositions of offspring in a further level (i.e., generation). Therefore, we will introduce in Chapter 3 a random m-spread model in which the probability distribution of the type composition of the offspring depends on the types of the ancestors in the past m generations in the family history. Figures 4, 5 and 6 give illustrations showing how a random 2-spread model with type set $\{a, b\}$ spreads. In Figure 4, we can see that one type a individual produces two type a individuals and this spread pattern represents a mode, called $\alpha^1_{i_1}$, in the random 2-spread model. After one generation when this population has produced its third generation, i.e., this 2-level tree $\alpha^1_{i_1}$ grows into a 3-level tree at time 1, it has a non-negative probability to grow into the tree $\alpha^2_{j_1}$ on the left or the tree $\alpha^2_{j_3}$ on the right or other possibilities, but it will never, i.e., with probability 0, grow into the tree $\alpha^2_{j_2}$ in the middle because $\alpha^2_{j_2}$ this tree does not share the same mode as the past two generations with $\alpha^1_{i_1}$. Also, Figure 6 shows how a 3-level tree grows into a 4-level tree in the random 2-spread model. In Chapter 3, we will give a formal construction of the m-dependent process and the random m-spread model and its related m-dependent process which describes how the tree (or population) grows and how the types are passed onto the next level (i.e., generation). We then introduce a corresponding branching process which is called the induced branching process to study the spread rate in the random m-spread model. Theorem 15 and Theorem 17 give results for the spread rates of a type and a pattern in the random m-spread model. Moreover, a comparison of two random m-spread models is given in Theorem 18 in an average sense.

Due to increasing interest in tree-indexed processes, many research works in physics, probability, dynamical systems and information theory have been done on studying the limit behaviors of the random fields on trees. For example, Berger and Ye Berger and Ye (1990) investigated the entropy rate for random fields on binary trees. Ye and Berger Ye and Berger (1996) proved the asymptotic equipartition property with convergence in probability for a PPG-invariant and ergodic random field on a homogeneous tree. Yang and Liu Yang and Liu (2002) studied a strong law of large numbers for the frequency of the occurrence of a state in the entire tree for a Markov chain field on a homogeneous tree. Yang and Ye Yang and Ye (2007) also proved the asymptotic equipartition property for nonhomogeneous Markov chains indexed by a homogeneous tree. In this paper, we take a different approach and obtain an extension on the frequency of the occurrence of a state within any given successive levels of the tree with the influence from not only the "parent" but also from the "ancestors in the most recent m levels".

Finally, Chapter 4 is devoted to the discussion of the relationship between the topological m-spread model and the random m-spread model, and in Chapter 5, we provide some examples and numerical evidence to support our main results in both models.

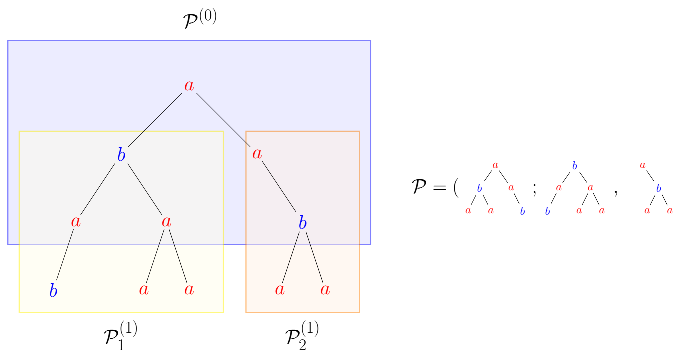


FIGURE 3. Parent and children in an m-pattern

2. Topological m-spread model

2.1. **Notations and setup.** Let $\mathcal{A} = \{a_i\}_{i=1}^k$ be a type set. For $d \in \mathbb{N}$, we denote by T_d the conventional d-tree, and define $\Delta_n = \{g \in T_d : 0 \leq |g| \leq n\}$, where $|g|$ stands for the length of g, i.e., the number of the edges from the root ϵ to g. For $m < n$ and $m, n \in \mathbb{N}$, define

$$\Delta_m^n = \Delta_n \backslash \Delta_m = \{g \in T_d : m < |g| \leq n\},$$

and for a finite set $F \subseteq T_d$ define $F_m^n = F \cap \Delta_m^n$. For finite set $F \subseteq \Delta_m$, $t \in F$ with $|t| = r$, and $n \in \mathbb{N}$ with $r + n \leq m$, we define $F_n^{(t)} = \{tg \in F : |tg| \leq n\}$ and $F_m^{(\epsilon)} = F$ clearly. Let $F \subseteq \Delta_m$ and assume at least one $g \in F$ such that $|g| = m$. A function $p : F \to \mathcal{A}$ is called an *m-order pattern* (or *m-pattern* for brevity), and $F = F_p$ is called the *support* of p, and we use the notation $\mathcal{P}_m$ to denote the set of the collection of m-patterns. For any pattern p, we also define $\Delta_m(p) = F_p \cap \Delta_m$ and $\Delta_m^n(p) = F_p \cap \Delta_m^n$. Supposing $p : F_p \to \mathcal{A}$ is a pattern on F_p, we denote by $p_g := p(g) \in \mathcal{A}$ for $g \in F_p$, the symbol at the coordinate g. Letting p be an m-pattern and F_p be its support, we decompose $F_p = \cup_{i=0}^m \Sigma_p^{(i)}$, where $\Sigma_p^{(0)} = \epsilon$ and $\Sigma_p^{(i)} = \{g \in F_p : |g| = i\} \subseteq F_p$ for $1 \leq i \leq m$, and define $\sigma_p := \{\sigma_p^{(i)}\}_{i=0}^m$, where $\sigma_p^{(i)} = \left|\Sigma_p^{(i)}\right|$, where $|A|$ stands for the cardinality of the set A. That is, $\Sigma_p^{(i)}$ is the ith layer vertices in F_p and $\sigma_p^{(i)}$ is its cardinality. Finally, we define $\underline{d}_p := \min_{0 \leq i \leq m} \sigma_p^{(i)}$ and $\overline{d}_p := \max_{0 \leq i \leq m} \sigma_p^{(i)}$.

2.1.1. *m-spread models and spread rate.* For $p \in \mathcal{P}_m$, we call $p^{(0)}$ the *parent* of p if $p^{(0)} \in \mathcal{P}_{m-1}$ and $p^{(0)} = p|_{(F_p)_{m-1}^{(\epsilon)}}$, where $p|_E$ is the *projection* of p on the finite set E, that is, $p|_E = \{p_g : g \in E\}$. In addition, for $1 \leq j \leq \sigma_p^{(1)}$, we define the j-th *child* with respect to p by $p_j^{(1)} = p|_{(F_p)_{m-1}^{(g_j)}}$ for $g_j \in \Sigma_p^{(1)}$. Hence, each $p \in \mathcal{P}_m$ can be written in the following form (cf. Figure 3)

$$p = (p^{(0)}; p_1^{(1)}, \ldots, p_{\sigma_p^{(1)}}^{(1)}). \tag{1}$$

Let $\mathcal{S} = \{p_i\}_{i=1}^l$ be a collection of m-patterns and suppose $\overline{d} := \max_{1 \leq i \leq l} \overline{d}_{p_i}$, $\mathcal{S}$ is called an *m-order spread model* (or *m-spread model* for brevity) if for any $p \in \mathcal{S}$ and $\forall 1 \leq j \leq \sigma_p^{(1)}$, there exists exactly one $q \in \mathcal{S}$ such that $p_j^{(1)} = q^{(0)}$.

Let $\mathcal{S}$ be an m-spread model and $1 \leq j \leq \sigma_p^{(1)}$. Since $p_j^{(1)}$ is parent of some $q(p_j^{(1)}) \in \mathcal{P}_m$, we replace $p_j^{(1)}$ by $q(p_j^{(1)})$ along each $1 \leq j \leq \sigma_p^{(1)}$. This generates a $(m+1)$-pattern, say τ_p^{m+1}. We continue the same process. Once the $\tau_p^{m+n} \in \mathcal{P}_{m+n}$ is constructed and $F_{\tau_p^{m+n}}$ is the support of τ_p^{m+n}. We replace each $p_j^{(1)} \in \mathcal{P}_{m-1}$ by $q(p_j^{(1)}) \in \mathcal{P}_m$ for all $1 \leq j \leq \sigma_{\tau_p^{m+n}}^{(n+1)}$ to generate $\tau_p^{m+n+1} \in \mathcal{P}_{m+n+1}$. Clearly, $\tau_p^{m+n}|_{\Delta_m(\tau_p)} = p$ for $n \geq 1$. Finally, we define

$$\tau_p := \lim_{n\to\infty} \tau_p^{m+n} \in \mathcal{A}^{T_{\overline{d}}}$$

and call it the *infinite spread pattern induced from p with respect to $\mathcal{S}$* (or *induced spread pattern from p*), and we denote by F_{τ_p} the support of τ_p.

Suppose $\mathcal{S}$ is a m-spread model. For $p \in \mathcal{S}$, let τ_p be the induced spread pattern from p with respect to $\mathcal{S}$. Denote by $\tau_p|_{\Delta_r^s}$ the projection of τ_p on Δ_r^s, for $r, s \in \mathbb{N}$. Let $\{k_n\}_{n=1}^\infty \subseteq \mathbb{N}$ be a sequence of natural numbers with $k_n \to \infty$ as $n \to \infty$ and define

$$s_n := \sum_{i=1}^n k_i. \tag{2}$$

Let $n \in \mathbb{N}$ and $\eta \in \mathcal{P}_n$ be any sub-pattern of τ_p for some $p \in \mathcal{S}$, i.e., $\eta = \tau_p|_F$, where F is a finite set of F_{τ_p}. We denote by $O_a(\eta)$ the number of occurrences of $a \in \mathcal{A}$ in η. The aim of this study is to compute the following spread rate of a in τ_p within the range $\Delta_{s_n}^{s_{n+1}}(\tau_p)$.

$$s_p(a; \{k_n\}_{n=1}^\infty) = \lim_{n\to\infty} s_p(a; [s_n, s_{n+1}]) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|},\ a \in \mathcal{A}, \tag{3}$$

whenever the limit (3) exists. In addition, for $k_n = k$, $\forall n \geq 1$ we are interested in the following spread rate within the fixed range $\Delta_{kn}^{k(n+1)}(\tau_p)$ as well.

$$s_p(a; k) = \lim_{n\to\infty} s_p(a; [kn, k(n+1)]) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{kn}^{k(n+1)}(\tau_p)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_p)\right|},\ a \in \mathcal{A}. \tag{4}$$

2.1.2. *Induced systems.* To compute the rates (3) and (4), a method which transforms a m-spread model to a 1-spread model (induced system) is proposed. Supposing $\mathcal{S}$ is a m-spread model over $\mathcal{A}$, we define the *induced type set* as

$$\mathbf{A} = \mathbf{A}(\mathcal{S}) = \{\mathbf{a} \in \mathcal{P}_{m-1} : \mathbf{a} \text{ is a parent or a child of } p \in \mathcal{S} \cap \mathcal{P}_m\},$$

on which an order is given, say $\mathbf{A} = \{\mathbf{a}_i\}_{i=1}^{\mathbf{k}}$. Therefore, the m-spread model can be transformed to the 1-spread model $\mathbf{S} = \mathbf{S}(\mathcal{S})$ over $\mathbf{A}$, as follows. For $p = (p^{(0)}; p_1^{(1)}, \ldots, p_{\sigma_p^{(1)}}^{(1)})$ (cf. (1)) and $\alpha \in \mathbf{A}$, which represents $p^{(0)}$, we write

$$\mathbf{p}_\alpha^{(1)} = \mathbf{p}_\alpha^{(1)}(p) = (\alpha; \alpha_1^{(1)} \cdots \alpha_{\sigma_\alpha^{(1)}}^{(1)}) := (p^{(0)}; p_1^{(1)} \cdots p_{\sigma_{p^{(0)}}^{(1)}}^{(1)}),$$

where $\alpha_j^{(1)} \in \mathbf{A}$ represents $p_j^{(0)}$ for $1 \leq j \leq \sigma_{p^{(0)}}^{(1)}$, and define

$$\Pi(\mathbf{p}_\alpha^{(1)}) = (\alpha_1^{(1)}, \ldots, \alpha_{\sigma_\alpha^{(1)}}^{(1)}).$$

Next, we replace each $\alpha_j^{(1)}$ by $\Pi(\mathbf{p}^{(1)}_{\alpha_j^{(1)}})$ for all $j=1,2,\ldots,\sigma_\alpha^1$ to obtain the pattern of the 2nd generation, i.e.,

$$\begin{aligned}\mathbf{p}_\alpha^{(2)} &= (\alpha;\Pi(\mathbf{p}^{(1)}_{\alpha_1^{(1)}})\Pi(\mathbf{p}^{(1)}_{\alpha_2^{(1)}})\cdots\Pi(\mathbf{p}^{(1)}_{\alpha^{(1)}_{\sigma_\alpha^{(1)}}}))\\ &= (\alpha;\alpha_1^{(2)}\cdots\alpha^{(2)}_{\sigma_\alpha^{(2)}}),\end{aligned}$$

where $\sigma_\alpha^{(2)}$ is the total number of the offspring in the 2nd generation. In this manner, once $\mathbf{p}_\alpha^{(n-1)}=(\alpha;\alpha_1^{(n-1)},\ldots,\alpha^{(n-1)}_{\sigma_\alpha^{(n-1)}})$ is defined, we denote the *offspring type chart of* α in the $(n-1)$th generation by

$$\Pi(\mathbf{p}_\alpha^{(n-1)})=\alpha_1^{(n-1)},\ldots,\alpha^{(n-1)}_{\sigma_\alpha^{(n-1)}}.$$

Then we replace offspring of type $\alpha_j^{(n-1)}$ in $\mathbf{p}_\alpha^{(n-1)}$ by $\Pi(\mathbf{p}^{(1)}_{\alpha_j^{(n-1)}})$, $j=1,2,\ldots,\sigma_\alpha^{(n-1)}$, to obtain $\mathbf{p}_\alpha^{(n)}$, the pattern of the nth generation and so on.

We call $\mathbf{S}=\mathbf{S}(\mathcal{S})=\{\mathbf{p}^{(1)}(p)\}_{p\in\mathcal{S}}=\{\mathbf{p}_\alpha^{(1)}\}_{\alpha\in\mathbf{A}}$ the 1-*spread model induced from* S (or *induced model*) over $\mathbf{A}$. Let $\mathbf{S}$ be the induced model from $\mathcal{S}$. For $\alpha,\beta\in\mathbf{A}$, denote by $\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta$ the number of the occurrences of β in $\Pi(\mathbf{p}_\alpha^{(n)})$. Let $\xi:\mathbf{A}\to\mathbf{A}^*$ be the associated substitution of $\mathbf{S}$, that is, $\xi(\alpha)=\Pi(\mathbf{p}_\alpha)$ for all $\alpha\in\mathbf{A}$ with length $|\xi(\alpha)|=\sigma_\alpha^{(1)}$. This substitution induces a morphism of the monoid $\mathbf{A}^*$ by assigning $\xi(\mathbf{B})=\xi(\mathbf{b}_0)\xi(\mathbf{b}_1)\cdots\xi(\mathbf{b}_n)$ if $\mathbf{B}=\mathbf{b}_0\mathbf{b}_1\cdots\mathbf{b}_n\in\mathbf{A}^*$ and $\xi(\mathbf{B})=\emptyset$ if $\mathbf{B}=\emptyset$. Denote the n-time iteration map of ξ as $\xi^n=\xi\circ\xi^{n-1}$. Proposition 3.1 Ban et al (2021) shows that the sets $\{\Pi(\mathbf{p}_\alpha^{(n)})\}_{\alpha\in\mathbf{A}}$ and $\{\xi^n(\alpha)\}_{\alpha\in\mathbf{A}}$ admit one-to-one correspondence. For the rest of our discussion of the induced model, two additional assumptions on the complexity of the spreading are made as in Ban et al (2021). That is, **1.** $\lim_{n\to\infty}|\xi(\alpha)|=\infty$ for every $\alpha\in\mathbf{A}$, and **2.** there exists an $\alpha_0\in\mathcal{A}$ such that $\xi(\alpha_0)$ begins with α_0. Given a substitution ξ and $\mathbf{A}=\{\mathbf{a}_i\}_{i=1}^{\mathbf{k}}$, we derive the associated ξ-matrix $\mathbf{M}=\mathbf{M}_\xi$, which is a $\mathbf{k}\times\mathbf{k}$ matrix defined by $\mathbf{M}=[\mathbf{m}_{ij}]:=[O_{\mathbf{a}_i}(\xi(\mathbf{a}_j))]$. In addition, the matrix $\mathbf{M}$ is assumed to be *irreducible*; that is, for every $1\le i,j\le\mathbf{k}$ there exists $n=n(i,j)$ such that $\mathbf{m}_{ij}^{(n)}>0$. It then follows from **1.** and **2.** that $\mathbf{M}$ is actually a *primitive* matrix (see Ban et al (2021) and (Queffélec, 2010, Proposition 5.5) for the equivalence), i.e., n can be chosen independent of i,j. Let ρ be the maximal eigenvalue of $\mathbf{M}$ and $\mathbf{v}=\mathbf{v}_\rho$ be the corresponding eigenvector. (Ban et al, 2021, Theorem 3.3) demonstrates that

$$s_\alpha(\beta)=\lim_{n\to\infty}s_\alpha^{(n)}(\beta):=\lim_{n\to\infty}\frac{\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta}{\sigma_\alpha^{(n)}}=\mathbf{v}(\beta).$$

Lemma 1 (Theorem 3.3. Ban et al (2021))**.** *Let* $\mathbf{S}$ *be the induced model from* S*, and* $\mathbf{A}$ *be the induced type set. Let* ξ*,* $\mathbf{M}$ *be the corresponding substitution map and* ξ*-matrix respectively. Suppose* ρ *is the spectral radius of* $\mathbf{M}$ *and* $\mathbf{v}$ *the corresponding eigenvector. The following statements hold.*

(i) *Let* $\alpha\in\mathbf{A}$*. Then the vector* $(s_\alpha(\beta))_{\beta\in\mathbf{A}}$ *is independent of* α *and* $(r_\alpha(\beta))_{\beta\in\mathbf{A}}=\mathbf{v}$*, i.e.,*

$$s_\alpha(\beta)=\lim_{n\to\infty}\frac{\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta}{\sigma_\alpha^{(n)}}=\mathbf{v}(\beta).$$

In addition, the speed of the convergence is geometric.

(ii) $c_1(\mathbf{v})\rho^m \leq \sum_{\alpha\in\mathbf{A}} s_\alpha^{(m)}(\beta) \leq c_2(\mathbf{v})\rho^m$ *for all* $m \in \mathbb{N}$ *and* $\beta \in \mathbf{A}$*, where*

$$c_1(\mathbf{v}) = \left(\frac{\min_{1\leq j\leq \mathbf{k}} \mathbf{v}(j)}{\max_{1\leq j\leq \mathbf{k}} \mathbf{v}(j)}\right) \text{ and } c_2 = \left(\frac{\max_{1\leq j\leq \mathbf{k}} \mathbf{v}(j)}{\min_{1\leq j\leq \mathbf{k}} \mathbf{v}(j)}\right).$$

Suppose $\mathbf{S} = \{\mathbf{p}_i\}_{i=1}^{\mathbf{l}}$ is induced from $\mathcal{S} = \{p_i\}_{i=1}^{l}$ over induced type set $\mathbf{A}$. For $a \in \mathcal{A}$ define

$$\theta(a) := \{\alpha \in \mathbf{A} : \alpha \text{ represents } p^{(0)} \in \mathcal{P}_{m-1} \text{ in which } p^{(0)}(\epsilon) = a\}.$$

2.2. **Main results.**

2.2.1. *Spread rate of a symbol.* Before presenting the main result, we provide a useful result for the study of the spread rate.

Lemma 2. *Let* $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ *be real sequences and* $\{c_n\}_{n=1}^{\infty}, \{d_n\}_{n=1}^{\infty}$ *be positive real sequences. Suppose*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{c_n}{d_n} = Q.$$

Then,

$$\lim_{n\to\infty} \frac{a_n + c_n}{b_n + d_n} = Q.$$

Furthermore, suppose that $\lim_{n\to\infty} \sum_{j=1}^{n} b_j = +\infty$*. Then,*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{n} a_j}{\sum_{j=1}^{n} b_j} = Q.$$

Proof. The equality $\lim_{n\to\infty} \frac{a_n+c_n}{b_n+d_n} = Q$ is immediate and we omit the proof. For the second part, we claim that for all $M > Q$ and $m < Q$ we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_n + \cdots + b_n} &\leq M, \text{ and}\\ \liminf_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_n + \cdots + b_n} &\geq m.\end{aligned}$$

Indeed, since there exists $N_1 \in \mathbb{N}$ such that $\frac{a_n}{b_n} < M$ for all $n \geq N_1$, we have, for all $n \geq N_1$, $a_n < Mb_n$ and

$$\begin{aligned}\frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &= \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + \frac{a_{N_1+1} + \cdots + a_n}{b_1 + \cdots + b_n}\\ &< \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + M\frac{b_{N_1+1} + \cdots + b_n}{b_1 + \cdots + b_n}\\ &< \frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + M.\end{aligned}$$

We note that N_1 is fixed and $\lim_{n\to\infty} \sum_{j=1}^{n} b_j = +\infty$. Therefore,

$$\limsup_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} \leq \limsup_{n\to\infty} \left(\frac{a_1 + \cdots + a_{N_1}}{b_1 + \cdots + b_n} + M\right) = M.$$

For the other inequality, since there exists $N_2 \in \mathbb{N}$ such that $\frac{a_n}{b_n} > m$ for all $n \leq N_2$, we derive by using the same argument as above that

$$\begin{aligned}\liminf_{n\to\infty} \frac{a_1 + \cdots + a_n}{b_1 + \cdots + b_n} &> \liminf_{n\to\infty} \left(\frac{a_1 + \cdots + a_{N_2}}{b_1 + \cdots + b_n} + m\frac{b_{N_2+1} + \cdots + b_n}{b_1 + \cdots + b_n}\right)\\ &= m.\end{aligned}$$

The proof is thus completed. □

Lemma 3. *Let* $\mathbf{S}$ *be the induced model from* S*, and* ρ *and* $\mathbf{v}$ *be defined as in Lemma 1. Suppose* $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ *is a sequence of natural numbers with* $k_n \to \infty$ *as* $n \to \infty$ *and* s_n *is defined in (2). Then for all* $\alpha, \beta \in \mathbf{A}$*, we have*

$$s_\alpha(\beta; \{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} \frac{O_\beta(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})}{\left|\Delta_{s_n+1}^{s_{n+1}}(\tau_\alpha)\right|} = \mathbf{v}(\beta). \tag{5}$$

In particular, if $k_n = k \ \forall n \in \mathbb{N}$*, we have*

$$s_\alpha(\beta; k) = \lim_{n\to\infty} \frac{O_\beta(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_\alpha)\right|} = \mathbf{v}(\beta). \tag{6}$$

Proof. **1.** It can be seen that (6) is not a direct consequence of (5) since $k_n = k \nrightarrow \infty$, and we provide the proof of (6) first. Note that

$$\left|\Delta_{kn}^{k(n+1)}(\tau_\alpha)\right| = \sigma_\alpha^{(kn+1)} + \cdots + \sigma_\alpha^{(k(n+1))}. \tag{7}$$

Lemma 1 is applied to show that

$$\lim_{n\to\infty} \frac{\left|\Pi(\mathbf{p}_\alpha^{(kn+j)})\right|_\beta}{\sigma_\alpha^{(kn+j)}} = \mathbf{v}(\beta) \text{ for } 1 \le j \le k. \tag{8}$$

Combining (8), (7) with the first result of Lemma 2, we have

$$\begin{aligned} s_\alpha(\beta; k) &= \lim_{n\to\infty} s_\alpha(\beta; \{k_n\}_{n=1}^{\infty}) \\ &= \lim_{n\to\infty} \frac{O_\beta(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_\alpha)\right|} \\ &= \lim_{n\to\infty} \frac{\left|\Pi(\mathbf{p}_\alpha^{(kn+1)})\right|_\beta + \cdots + \left|\Pi(\mathbf{p}_\alpha^{(k(n+1))})\right|_\beta}{\sigma_\alpha^{(kn+1)} + \cdots + \sigma_\alpha^{(k(n+1))}} \\ &= \mathbf{v}(\beta). \end{aligned}$$

This completes the proof of (6).

2. By an argument similar to the preceding, the second result of Lemma 2 yields (5). □

Theorem 4 below provides a formula for the spread rate of an $a \in \mathcal{A}$, and it reveals that the first goal stated in the introduction is fully achieved.

Theorem 4. *Let* $\mathcal{S} = \{p_j\}_{j=1}^{l}$ *be a* m*-spread model over* $\mathcal{A}$*, and* $\mathbf{S} = \{\mathbf{p}_i\}_{i=1}^{\mathbf{l}}$ *be the associated induced* 1*-order spread model over the induced type set* $\mathbf{A}$*. Then, the following statements hold.*

(i) *Suppose* $k \in \mathbb{N}$*, for* $a \in \mathcal{A}$ *and* $p \in \mathcal{S}$ *we have*

$$s_p(a; k) = \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{kn}^{k(n+1)}(\tau_p)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_p)\right|} = \sum_{\beta\in\theta(a)} \mathbf{v}(\beta). \tag{9}$$

(ii) *Let $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be an increasing sequence with $k_n \to \infty$ as $n \to \infty$. Supposing $a \in \mathcal{A}$ and $p \in S$, we have*

$$
\begin{aligned}
s_p(a; \{k_n\}_{n=1}^{\infty}) &= \lim_{n\to\infty} s_p(a, [s_n, s_{n+1}]) \\
&= \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|} = \sum_{\beta\in\theta(a)} \mathbf{v}(\beta). \qquad (10)
\end{aligned}
$$

Proof. **1.** We prove the equality (10) first. Let τ_p be the induced spread pattern from $p \in \mathcal{S}$. Suppose

$$F_{\tau_p} = \cup_{i=0}^{\infty} \Sigma_{\tau_p}^{(i)}$$

is the decomposition of the support of τ_p. For $n \in \mathbb{N}$, supposing α represents $p^{(0)}$ in $\mathbf{A}$, we have

$$
\begin{aligned}
O_a(\tau_p|_{\Sigma_p^{(n)}}) &= \sum_{q\in S,\ q(\epsilon)=a} \text{number of occurrences of } q \text{ in } \tau_p \text{ with root in } \Sigma_p^{(n)} \\
&= \sum_{\beta\in\theta(a)} O_\beta(\tau_p|_{\Delta_{n-1}^{n+m-1}(\tau_p)}) \qquad (11) \\
&= \sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta .
\end{aligned}
$$

The equality (11) holds due to the fact that once q appears in $\Delta_{n-1}^{n+m-1}(\tau_p)$ and $q(\epsilon) = a$, then q contributes an $a's$ in the nth layer $\Sigma_{\tau_p}^{(n)}$ in F_{τ_p}. Therefore, supposing $\alpha \in \mathbf{A}$ represents $p^{(0)}$, we have

$$
\begin{aligned}
s_p(a; \{k_n\}_{n=1}^{\infty}) &= \lim_{n\to\infty} s_p(a, [s_n, s_{n+1}]) \\
&= \lim_{n\to\infty} \frac{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_p)\right|} \\
&= \lim_{n\to\infty} \frac{\sum_{r=s_n+1}^{s_{n+1}} \sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(r)})\right|_\beta}{\sigma_\alpha^{(s_n+1)} + \cdots + \sigma_\alpha^{(s_{n+1})}} \\
&= \sum_{\beta\in\theta(a)} \lim_{n\to\infty} \frac{\sum_{r=s_n+1}^{s_{n+1}} \left|\Pi(\mathbf{p}_\alpha^{(r)})\right|_\beta}{\sum_{r=s_n+1}^{s_{n+1}} \sigma_\alpha^{(r)}} \\
&= \sum_{\beta\in\theta(a)} \mathbf{v}(\beta).
\end{aligned}
$$

The last equality follows from Lemma 3 and Lemma 2.

2. Then the equality (9) follows from the same argument as above and the second part of Lemma 3. This completes the proof. □

2.2.2. *Spread rate of a pattern.* Given an m-spread model $\mathcal{S}$ over $\mathcal{A}$, and $\{k_n\}_{n=1}^{\infty}$ is a increasing sequence of natural numbers, Theorem 4 unveils that we can compute the spread rate of a symbol $a \in \mathcal{A}$ within the range $\Delta_{s_n}^{s_{n+1}}$, where $\{s_n\}_{n=1}^{\infty}$ is defined in (2). For a pattern η appearing in a $p \in \mathcal{S}$, that is, where η is a subpattern of p, the method developed in Theorem 4 also allows us to compute the spread rate of the pattern η. Define

$$\mathcal{S}^{(0)} = \{p^0 \in \mathcal{P}_{m-1} : p^{(0)} \text{ is parent of a } p \in \mathcal{S}\}.$$

For a natural number r with $r \leq m-1$, we say a pattern $\eta \in \mathcal{P}_r$ *appears* in some $p^{(0)} \in \mathcal{P}_{m-1}$ if $p^{(0)}|_{F_p \cap \Delta_r} = \eta$; that is, η is a subpattern of $p^{(0)}$ on $F_p \cap \Delta_r$. The aim of this section is to compute the spread rate

$$s_p(\eta; \{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} s_p(\eta; [s_n, s_{n+1}]) := \lim_{n\to\infty} \frac{O_\eta(\tau_p|_{\Delta_{s_n}^{s_{n+1-r}}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1-r}}(\tau_p)\right|}. \tag{12}$$

Define

$$\mathcal{A}_r = \{\widetilde{\eta} \in \mathcal{P}_r : \widetilde{\eta} \text{ appears in some } p^{(0)} \in \mathcal{S}^{(0)}\}.$$

We construct a new spread model over $\mathcal{A}_r$, say $\mathcal{S}_r$, in the following. Letting $p \in \mathcal{S}$ $(= \mathcal{S}_m)$ and $F = F_p$ be the support of the pattern p, we define a pattern $\widetilde{p} \in \mathcal{P}_{m-r}^r \cap \mathcal{A}_r^{F_{m-r}}$ as follows, where $\mathcal{P}_{m-r}^r$ is the set of all $(m-r)$-patterns in which each symbol is an r-pattern in $\mathcal{P}_r$. For all $g \in F_{m-r}$ if $p|_{F_r^{(g)}} = q \in \mathcal{A}_r$, we then define $\widetilde{p}_g = q$, i.e., we replace every subpattern of p in $F_r^{(g)}$ for all $g \in F_{m-r}$ by a new symbol $q \in \mathcal{A}_r$. Note that $\mathcal{S}$ is an m-spread model and that $\widetilde{p} = \widetilde{p}(p)$ is well-defined since every subpattern $p|_{F_r^{(g)}}$ $\forall g \in F_{m-r}$ appears in some $p^{(0)} \in \mathcal{S}^{(0)}$. Let $\widetilde{p}$ be the symbol in $\mathcal{A}_r$ representing p, then define an $(m-r)$-spread model with respect to $\mathcal{S}$ by $\mathcal{S}_r = \{\widetilde{p}(p)\}_{p\in\mathcal{S}}$. Theorem 5 below shows that the new $(m-r)$-spread model help us to compute the spread rate of the pattern $\eta \in \mathcal{P}_r$.

Theorem 5. *Let $\mathcal{S}$ be an m-spread model over $\mathcal{A}$ and $\{k_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$ be an increasing sequence with $k_n \to \infty$ as $n \to \infty$. For $r \leq m-1$, suppose $\mathcal{S}_r$ is the $(m-r)$-spread model with respect to $\mathcal{S}$ over $\mathcal{A}_r$. Then, for any $\eta \in \mathcal{P}_r$, we have*

$$s_p(\eta; \{k_n\}_{n=1}^{\infty}) = s_{\widetilde{p}}(\widetilde{\eta}; \{k_n\}_{n=1}^{\infty}), \tag{13}$$

where $\widetilde{p}$ (resp. $\widetilde{\eta}$) is the pattern (resp. symbol) in $\mathcal{P}_{m-r}^r$ (resp. $\mathcal{A}_r$) representing p (resp. η). Furthermore, (13) is also valid for the case where $k_n = k$ $\forall n \in \mathbb{N}$.

Proof. It is sufficient to prove the case where $k_n \to \infty$ as $n \to \infty$ and the other case can be treated in the same fashion. According to definition (12), for every $p \in \mathcal{S}$, there is a pattern $\widetilde{p} = \widetilde{p}(p) \in \mathcal{A}_r^{F_{m-r}} \cap \mathcal{P}_{m-r}^r$ representing p. Let τ_p (resp. $\tau_{\widetilde{p}}$) be the infinite spread pattern induced from p (resp. $\widetilde{p}$) with respect to $\mathcal{S}$ (resp. $\mathcal{S}_r$). Suppose F_{τ_p} (resp. $F_{\tau_{\widetilde{p}}}$) is the support of τ_p (resp. $\tau_{\widetilde{p}}$). It can be easily checked that $F_{\tau_p} = F_{\tau_{\widetilde{p}}}$. Supposing $F_{\tau_p} = \cup_{i=1}^{\infty} \Sigma_{\tau_p}^{(i)}$ $(F_{\tau_{\widetilde{p}}} = \cup_{i=1}^{\infty} \Sigma_{\tau_{\widetilde{p}}}^{(i)})$, it follows from the definitions of the spread mode and $\mathcal{S}_r$, we have

$$O_\eta(\tau_p|_{\Delta_{i-1}^{i+r-1}(\tau_p)}) = O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Sigma_{\widetilde{p}}^{(i)}}) \text{ for } 0 \leq i \in \mathbb{N}.$$

Therefore,

$$\begin{aligned}
s_p(\eta;\{k_n\}_{n=1}^{\infty}) &= \lim_{n\to\infty} s_p(\eta;[s_n,s_{n+1}-r]) \\
&= \lim_{n\to\infty} \frac{O_\eta(\tau_p|_{\Delta_{s_n}^{s_{n+1}-r}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_p)\right|} \\
&= \lim_{n\to\infty} \frac{O_\eta(\tau_p|_{\Delta_{s_n}^{s_n+r}(\tau_p)}) + O_\eta(\tau_p|_{\Delta_{s_n+1}^{s_n+1+r}(\tau_p)}) + \cdots + O_\eta(\tau_p|_{\Delta_{s_{n+1}}^{s_{n+1}+r}(\tau_p)})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_p)\right|} \\
&= \lim_{n\to\infty} \frac{O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Sigma_{\widetilde{p}}^{(s_n+1)}}) + O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Sigma_{\widetilde{p}}^{(s_n+2)}}) + \cdots + O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Sigma_{\widetilde{p}}^{(s_{n+1}+1)}})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_p)\right|} \\
&= \lim_{n\to\infty} \frac{O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\widetilde{p}})})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_p)\right|} \\
&= \lim_{n\to\infty} \frac{O_{\widetilde{\eta}}(\tau_{\widetilde{p}}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\widetilde{p}})})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_{\widetilde{p}})\right|} \\
&= s_{\widetilde{p}}(\widetilde{\eta};\{k_n\}_{n=1}^{\infty}).
\end{aligned}$$

This completes the proof. □

2.2.3. *Comparison of two spread models.* In this section, we discuss the spread rate of an $a \in \mathcal{A}$ of two topological spread models so as to achieve the second goal stated in the introduction.

Let $\mathcal{A}$ be a type set, and $\mathcal{S}$ and $\mathcal{S}'$ be two m-spread models over $\mathcal{A}$. Suppose $\mathbf{S}$ and $\mathbf{S}'$ are the corresponding induced 1-spread models of $\mathcal{S}$ and $\mathcal{S}'$ over $\mathbf{A}$ and $\mathbf{A}'$ respectively. Write $\mathbf{d} = \min_{\alpha\in\mathbf{A}} \sigma_\alpha^{(1)}$ and $\mathbf{D} = \max_{\alpha\in\mathbf{A}} \sigma_\alpha^{(1)}$ for $\mathbf{S}$, and $\mathbf{d}'$ and $\mathbf{D}'$ for $\mathbf{S}'$ in a similar manner.

Lemma 6 (Theorem 4.5.12 Lind and Marcus (1995))**.** *Let A be a primitive matrix with Perron eigenvalue ρ. Let v, w be right, left Perron eigenvectors of A, i.e., vectors v, $w > 0$ such that $Av = \rho v$, that $wA = \rho w$, and that $w \cdot v = 1$. Then, for each i, j,*

$$A_{i,j}^n = [(v(i)w(j)) + \lambda_{ij}(n)]\rho^n,$$

where $\lambda_{ij}(n) \to 0$ as $n \to \infty$.

Lemma 7 (Theorem 4.4.7 Lind and Marcus (1995))**.** *Let A be an irreducible matrix, $0 \le B \le A$, and $B_{kl} < A_{kl}$ for a pair k, $l$of indices. Then $\rho_B \le \rho_A$.*

Theorem 8. *Suppose that S and S' are two m-spread models over $\mathcal{A}$, that $\mathbf{S}$, $\mathbf{S}'$, $\mathbf{d}$, $\mathbf{D}$, $\mathbf{d}'$, $\mathbf{D}'$, $\mathbf{A}$, $\mathbf{A}'$ are defined as above, and that $\mathbf{A} = \mathbf{A}'$.*

(i) *If $\mathbf{D}' < \mathbf{d}$, then for any $\{k_n\}_{n=1}^{\infty}$ with $k_n \to \infty$, $p \in S$ and $a \in \mathcal{A}$ we have*

$$\lim_{n\to\infty} \frac{O_a(\tau_p'|_{\Delta_{s_n}^{s_{n+1}}(\tau_p')})}{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})} = 0, \tag{14}$$

where the sequence $\{s_n\}_{n=1}^{\infty}$ is defined in (2).

(ii) *Under the same condition of (i), for all $p \in S$, $a \in \mathcal{A}$ and $k \in \mathbb{N}$, we have*

$$\lim_{n\to\infty} \frac{O_a(\tau'_p|_{\Delta_{kn}^{k(n+1)}(\tau'_p)})}{O_a(\tau_p|_{\Delta_{kn}^{k(n+1)}(\tau_p)})} = 0. \tag{15}$$

(iii) *If* $\left|\Pi(\mathbf{p}_\alpha^{(1)\prime})\right|_\beta \leq \left|\Pi(\mathbf{p}_\alpha^{(1)})\right|_\beta$ *and there exists a pair* $(\gamma, \delta) \in \mathbf{A}^2$ *such that* $\left|\Pi(\mathbf{p}_\gamma^{(1)\prime})\right|_\delta < \left|\Pi(\mathbf{p}_\gamma^{(1)})\right|_\delta$*, then for any* $\{k_n\}_{n=1}^\infty$ *with* $k_n \to \infty$*,* $p \in S$ *and* $a \in \mathcal{A}$*, the equality (14) holds.*

(iv) *Under the same condition of (iii), for all $p \in S$, $a \in \mathcal{A}$ and $k \in \mathbb{N}$, the equality (15) holds.*

Proof. **1.** It follows from the same argument in the proof of Theorem 3.4 Ban et al (2021) that we have $\rho' < \rho$.

Let $\mathbf{M} = [\mathbf{m}_{\gamma,\delta}]$, $\mathbf{M}' = [\mathbf{m}'_{\gamma,\delta}]$ be the corresponding ξ-matrices of the 1-spread models induced from $\mathbf{S}$ and $\mathbf{S}'$. Denote $\mathbf{M}^n = [\mathbf{m}_{\gamma,\delta}^{[n]}]$ and $\mathbf{M}'^n = [\mathbf{m}'^{[n]}_{\gamma,\delta}]$ for $n \in \mathbb{N}$. Then, for $\alpha \in \mathbf{A}$, we have

$$\begin{aligned}
\sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta &= \sum_{\alpha\in\mathbf{A}} \mathbf{m}'^{[n]}_{\beta,\alpha} \leq c_2(\mathbf{v}')\,(\rho')^n \\
&= \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right)\left(\frac{\rho'}{\rho}\right)^n c_1(\mathbf{v})\rho^n \\
&\leq \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right)\left(\frac{\rho'}{\rho}\right)^n \sum_{\alpha\in\mathbf{A}} \mathbf{m}^{[n]}_{\beta,\alpha} \\
&\leq \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right)\left(\frac{\rho'}{\rho}\right)^n \sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta .
\end{aligned}$$

Therefore,

$$\frac{\sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \leq \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right)\left(\frac{\rho'}{\rho}\right)^n. \tag{16}$$

It follows from (16) and from Lemma 6 that for all α**,** β, α', $\beta' \in \mathbf{A}$ we have

$$\frac{\left|\Pi(\mathbf{p}_{\alpha'}^{(n)\prime})\right|_{\beta'}}{\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \leq (c_3+\lambda(n))\frac{\sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\sum_{\alpha\in\mathbf{A}} \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \leq (c_3+\lambda_1(n))\left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right)\left(\frac{\rho'}{\rho}\right)^n, \tag{17}$$

where c_3 is a constant and $\lambda_1(n) \to 0$ as $n \to \infty$. Suppose $F_{\tau_p} = \cup_{i=0}^\infty \Sigma_{\tau_p}^{(i)}$ is the decomposition of the support of τ_p. It follows from Lemma 6 again that there exists c_4, c_5, $\lambda_2(n)$ and $\lambda_3(n)$ with $\lambda_2(n) \to 0$, $\lambda_3(n) \to 0$ as $n \to \infty$ such that for any α_1, β_1, α_2, $\beta_2 \in \mathbf{A}$,

$$\frac{\sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} = \frac{(c_4+\lambda_2(n))\left|\Pi(\mathbf{p}_{\alpha_1}^{(n)\prime})\right|_{\beta_1}}{(c_5+\lambda_3(n))\left|\Pi(\mathbf{p}_{\alpha_2}^{(n)})\right|_{\beta_2}}. \tag{18}$$

Combining (17) with (18) yields

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \frac{O_a(\tau_p'|_{\Delta_{s_n}^{s_{n+1}}(\tau_p')})}{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})} = \lim_{n\to\infty} \frac{\sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\sum_{\beta\in\theta(a)} \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \\
&= \lim_{n\to\infty} \frac{(c_4+\lambda_2(n)) \left|\Pi(\mathbf{p}_{\alpha_1}^{(n)\prime})\right|_{\beta_1}}{(c_5+\lambda_3(n)) \left|\Pi(\mathbf{p}_{\alpha_2}^{(n)})\right|_{\beta_2}} \\
&= \lim_{n\to\infty} \frac{(c_4+\lambda_2(n))}{(c_5+\lambda_3(n))} (c_3+\lambda_1(n)) \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right) \left(\frac{\rho'}{\rho}\right)^n \\
&= 0.
\end{aligned}$$

This completes the proof.

2. The proof is almost identical to the proof of (i) of Theorem 8, so we omit it.

3. We note that the assumption of $\left|\Pi(\mathbf{p}_\alpha^{(1)\prime})\right|_\beta \leq \left|\Pi(\mathbf{p}_\alpha^{(1)})\right|_\beta$ for all α, $\beta \in \mathbf{A}$ implies $\mathbf{M}' \leq \mathbf{M}$. Furthermore, the condition $\left|\Pi(\mathbf{p}_\gamma^{(1)\prime})\right|_\delta < \left|\Pi(\mathbf{p}_\gamma^{(1)})\right|_\delta$ infers that $\mathbf{m}'_{\gamma,\delta} < \mathbf{m}_{\gamma,\delta}$ for indices γ, $\delta \in \mathbf{A}$. Hence, Lemma 7 is applied to show that $\rho' < \rho$. On the other hand,

$$\begin{aligned}
\left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta &= \mathbf{m}'^{[n]}_{\beta,\alpha} \leq c_2(\mathbf{v}')(\rho')^n = \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right) \left(\frac{\rho'}{\rho}\right)^n c_1(\mathbf{v})\rho^n \\
&\leq \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right) \left(\frac{\rho'}{\rho}\right)^n \mathbf{m}^{[n]}_{\beta,\alpha} \\
&= \left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta .
\end{aligned}$$

Thus,

$$\frac{\left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \leq \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right) \left(\frac{\rho'}{\rho}\right)^n .$$

Using the same argument as above, we have

$$\begin{aligned}
0 &\leq \lim_{n\to\infty} \frac{O_a(\tau_p'|_{\Delta_{s_n}^{s_{n+1}}(\tau_p')})}{O_a(\tau_p|_{\Delta_{s_n}^{s_{n+1}}(\tau_p)})} = \lim_{n\to\infty} \frac{\left|\Pi(\mathbf{p}_\alpha^{(n)\prime})\right|_\beta}{\left|\Pi(\mathbf{p}_\alpha^{(n)})\right|_\beta} \\
&\leq \lim_{n\to\infty} \left(\frac{c_2(\mathbf{v}')}{c_1(\mathbf{v})}\right) \left(\frac{\rho'}{\rho}\right)^n \\
&= 0.
\end{aligned}$$

This completes the proof.

4. The proof follows the same method as in the proof of (iii), so we omit it. □

Remark 9. *Theorem 8 allows one to show, by comparing ξ-matrices $\mathbf{M}$ and $\mathbf{M}'$, the numbers of occurrences in one model would dominate those in another model. More specifically, Theorem 8 (i) implies that the minimal column sum of $\mathbf{M}$ is strictly larger than the maximal column sum of $\mathbf{M}'$, while Theorem 8 (iii) require* *only* $\mathbf{M} \geq \mathbf{M}'$ *entrywise with a strictly larger entry. These concrete criteria can then be put into practical applications.*

3. Random m-spread models

3.1. **Notations and setup.** In this section, we consider m as a fixed natural number and adopt the notations defined in Section 2.2 such as $\mathcal{A}$, T_b, Δ_n, Δ_m^n, $\Delta_m^n(\tau_\alpha)$, $|g|$, $\mathcal{P}_n$, and $\alpha^{(0)}$, $\Sigma_\alpha^{(r)}$, $\sigma_\alpha^{(1)}$ for $\alpha \in \mathcal{P}_m$, etc.

In addition, for any n and each n-pattern $\alpha = \alpha_i^n \in \mathcal{P}_n$ introduced in Section 2.1, we also define the following:

(i) When $\alpha = \alpha_i^n \in \mathcal{P}_n$, we write $(\alpha_i^n)^{(0)}$ for $\alpha^{(0)}$.

(ii) We write $\Sigma_\alpha^{(r)} = \{g_1^{\alpha,r}, g_2^{\alpha,r}, \cdots, g_{\sigma_\alpha^{(r)}}^{\alpha,r}\}$.

(iii) For each $r = 1, 2, \cdots, n-1$, $j = 1, 2, \cdots, \sigma_\alpha^{(r)}$, let $\alpha_j^{(r)}$ $(=(\alpha_i^n)_j^{(r)})$ be the subtree rooted at $g_j^{\alpha,r}$ in the rth level of α and up to the nth level in α.

(iv) Let $\epsilon(\alpha)$ be the root of α.

(1) Each m-pattern $\alpha \in \mathcal{P}_m$ is determined by its parent $\alpha^{(0)}$ and its children $\alpha_j^{(1)}$, $j = 1, 2, \cdots, \sigma_\alpha^{(1)}$. Denoted this pattern by

$$\alpha = (\alpha^{(0)}; \alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_{\sigma_\alpha^{(1)}}^{(1)}).$$

We will also use the similar notations $\sigma_\zeta^{(r)}$, $\zeta^{(0)}$, and $\zeta_j^{(r)}$, etc. for the random pattern ζ defined later in this section and, in this case, these quantities will become random elements.

3.1.1. *The random m-patterns and the random m-spread model.* Let $\mathcal{Q}_m$ be a subset of $\mathcal{P}_m$ such that, for every $\beta \in \mathcal{Q}_m$, there exists $q \in \mathcal{Q}_m$ such that

$$\beta_j^{(1)} = q^{(0)} \qquad \text{for all } j = 1, 2, \cdots, \sigma_\beta^{(1)}.$$

Since $\mathcal{P}_m$ is finite, so is $\mathcal{Q}_m$. Let $\mathcal{Q}_m = \{\alpha_1^m, \cdots, \alpha_{l_m}^m\}$.

Let $\mathcal{Q}_m^{(0)} = \{\beta^{(0)} : \beta \in \mathcal{Q}_m\}$ be the collection of the parents of all m-patterns in $\mathcal{Q}_m$. We call $\mathcal{Q}_m^{(0)}$ the parent set of $\mathcal{Q}_m$.

Let $f : \mathcal{Q}_m^{(0)} \times \mathcal{Q}_m \to [0, 1]$ be a function such that

(i) for any $\alpha \in \mathcal{Q}_m^{(0)}$ and any $\beta \in \mathcal{Q}_m$, if $\beta^{(0)} \neq \alpha$, then $f(\alpha, \beta) = 0$;

(ii) for each $\alpha \in \mathcal{Q}_m^{(0)}$, $\sum_{\beta \in \mathcal{Q}_m} f(\alpha, \beta) = 1$.

So, for any fixed $\alpha \in \mathcal{Q}_m^{(0)}$, $f(\alpha, \cdot) \equiv \{f(\alpha, \beta) : \beta \in \mathcal{Q}_m\}$ forms a probability distribution. Each $f(\alpha, \beta)$ can be considered as the probability for the $(m-1)$-pattern α to grow into the m-pattern β. Therefore, by Kolmogorov extension theorem, we can construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and random elements $\{\zeta_\alpha : \alpha \in \mathcal{Q}_m^{(0)}\}$ such that, for each α, ζ_α is a $\mathcal{Q}_m$-valued random element on $(\Omega, \mathcal{F}, \mathbb{P})$ such that $(\zeta_\alpha)^{(0)}(\omega) = \alpha$ for each $\omega \in \Omega$ and the probability of ζ_α taking value at $\beta \in \mathcal{Q}_m$ is $f(\alpha, \beta)$. Such a random element ζ_α is called a *random m-pattern* with pattern distribution $f(\alpha, \cdot)$ on $\mathcal{Q}_m^{(0)} \times \mathcal{Q}_m$.

The nonempty collection $\mathcal{R} = \{\zeta_\alpha : \alpha \in \mathcal{Q}_m^{(0)}\}$ of random m-patterns is called the *random m-spread model* over the type set $\mathcal{A}$ with pattern distribution f on $\mathcal{Q}_m^{(0)} \times \mathcal{Q}_m$. Note that, if $\zeta_\alpha \in \mathcal{R}$ and $f(\alpha, \beta) > 0$ for $\beta \in \mathcal{Q}_m$, then for all $j = 1, 2, \cdots, \sigma_\beta^{(1)}$, $\zeta_{\beta_j^{(1)}} \in \mathcal{R}$.

Remark 10. *The domain $\mathcal{Q}_m^{(0)} \times \mathcal{Q}_m$ of the pattern distribution function f is chosen to make each $(m-1)$-pattern in the tree to grow into an m-pattern with probability one. In this case, the tree will grow continuously. However, the function f can be defined on $\mathcal{P}_{m-1} \times \mathcal{P}_m$ as a generalization of our model. Similar discussions and proofs in this paper can be adapted by adding a "pattern" to represent the possible situation in which the $(m-1)$-pattern does not grow at all and the corresponding subtree terminates.*

For any $\alpha \in \mathcal{Q}_m^{(0)}$ such that $\zeta_\alpha \in \mathcal{R}$, let $\tau_\alpha^{m-1} = \alpha$ and we will construct a sequence $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ of random elements on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and each realization of $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ can be considered as a growing tree and can be used to describe the evolution of the type structure of the population over time. Here, τ_α^{m-1} represents the initial type structure for the population from time (i.e., generation) 0 up to time (i.e., generation) $m-1$.

For $n = 0$, since $\alpha \in \mathcal{Q}_m^{(0)}$, it is the parent of some m-pattern in $\mathcal{Q}_m$. So, we can replace α with an m-pattern $\alpha_i^m \in \mathcal{Q}_m$ to obtain a random element τ_α^m on $(\Omega, \mathcal{F}, \mathbb{P})$ with probability

$$\mathbb{P}(\tau_\alpha^m = \alpha_i^m) = f(\alpha, \alpha_i^m)$$

for all $i = 1, 2, \cdots, l_m$. Note that τ_α^m has the same distribution as the random m-pattern ζ_α.

For $n = 1$, if $\tau_\alpha^m = \alpha_{i_0}^m$, then $(\alpha_{i_0}^m)_j^{(1)} \in \mathcal{Q}_m^{(0)}$ for $j = 1, 2, \cdots, \sigma^{(1)}_{\alpha_{i_0}^m}$ and so we can replace each of them with some $\alpha_{i_j}^m \in \mathcal{Q}_m$ to obtain a $(m+1)$-pattern $\alpha_i^{m+1} \in \mathcal{P}_{m+1}$ as the value of τ_α^{m+1} with probability

$$\begin{aligned}
&\mathbb{P}(\tau_\alpha^{m+1} = \alpha_i^{m+1} | \tau_\alpha^m = \alpha_{i_0}^m) \\
= &\prod_{j=1}^{\sigma^{(1)}_{\alpha_{i_0}^m}} \mathbb{P}\big((\tau_\alpha^{m+1})_j^{(1)} = \alpha_{i_j}^m | \tau_\alpha^m = \alpha_{i_0}^m\big) \\
= &\prod_{j=1}^{\sigma^{(1)}_{\alpha_{i_0}^m}} f\big((\alpha_{i_0}^m)_j^{(1)}, \alpha_{i_j}^m\big)
\end{aligned}$$

with the assumption that the replacements of each $(m-1)$-pattern with an m-pattern is independent of the simultaneous replacement of other $(m-1)$-patterns. It should be noted that this probability is possibly positive only when $(\alpha_{i_j}^m)^{(0)} = (\alpha_{i_0}^m)_j^{(1)}$ for all j. Hence, the type structure in the first $m+1$ generations of τ_α^m and τ_α^{m+1} are identical with probability 1. Therefore, we can consider that τ_α^m is growing into τ_α^{m+1}. Here, τ_α^{m+1} is a random element taking values in $\mathcal{P}_{m+1}$.

Figures 4 and 5 are examples, with $m = 2$, which illustrate how 1-patterns $\alpha_{i_1}^1$ and $\alpha_{i_2}^1$ grow into 2-patterns $\alpha_{j_1}^2, \alpha_{j_2}^2, \alpha_{j_3}^2$ and $\alpha_{j_4}^2$ with respect to the corresponding probabilities. Figure 6 gives some ideas about a 2-pattern growing into a 3-pattern.

Assume that τ_α^{m+n} is constructed and note that τ_α^{m+n} takes values in $\mathcal{P}_{m+n}$. Given $\tau_\alpha^{m+n} = \alpha_{i_0}^{m+n} \in \mathcal{P}_{m+n}$, we then replace each $(m-1)$-pattern $(\alpha_{i_0}^{m+n})_j^{(n+1)} \in \mathcal{Q}_m^{(0)}$ rooted at $g_j^{\alpha_{i_0}^{m+n}, n+1}$, $j = 1, 2, \cdots, \sigma^{(n)}_{\alpha_{i_0}^{m+n}}$, with some $\alpha_{i_j}^m \in \mathcal{Q}_m$ to obtain a

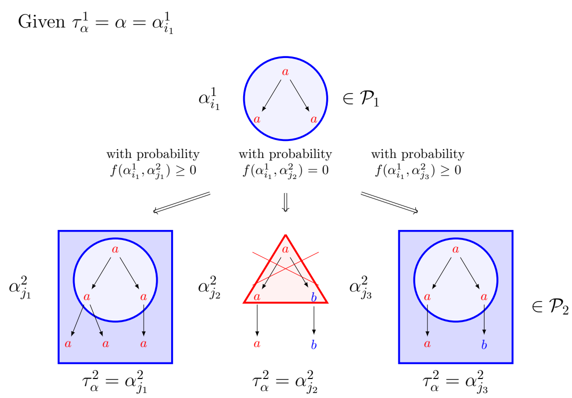


FIGURE 4. Illustration of a 1-spread pattern $\alpha_{i_1}^1$ growing into a 2-spread pattern

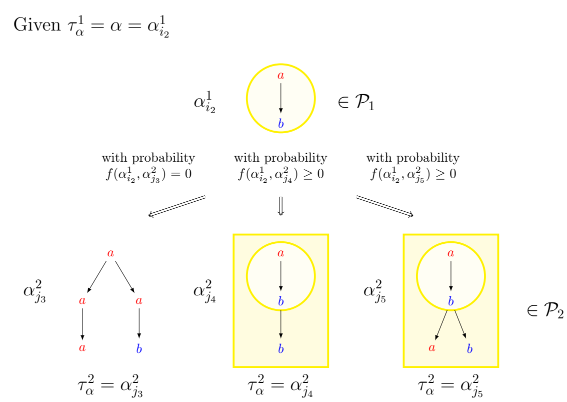


FIGURE 5. Illustration of a 1-spread pattern $\alpha_{i_2}^1$ growing into a 2-spread pattern

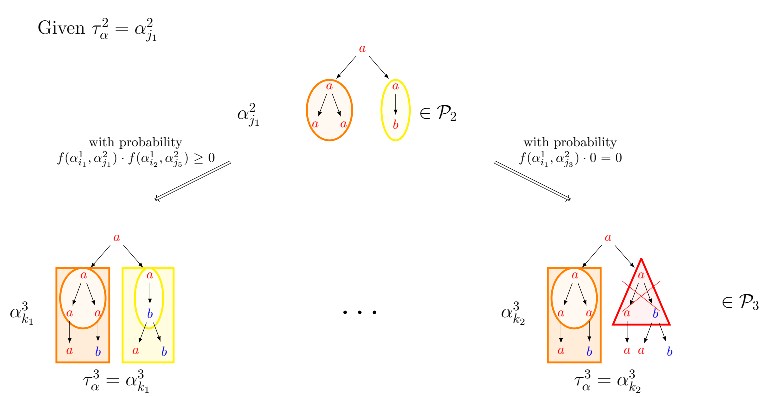


FIGURE 6. Illustration of a 2-spread pattern $\alpha_{j_1}^2$ growing into a 3-spread pattern

$(m+n+1)$-pattern α_i^{m+n+1} as the value of τ_α^{m+n+1} with probability

$$\begin{aligned}
&\mathbb{P}(\tau_\alpha^{m+n+1}=\alpha_i^{m+n+1}|\tau_\alpha^{m+n}=\alpha_{i_0}^{m+n}) \\
= &\prod_{j=1}^{\sigma^{(n+1)}_{\alpha_{i_0}^{m+n}}} \mathbb{P}\big((\tau_\alpha^{m+n+1})_j^{(n+1)}=\alpha_{i_j}^m|\tau_\alpha^{m+n}=\alpha_{i_0}^{m+n}\big) \\
= &\prod_{j=1}^{\sigma^{(n+1)}_{\alpha_{i_0}^{m+n}}} f\big((\alpha_{i_0}^{m+n})_j^{(n+1)},\alpha_{i_j}^m\big),
\end{aligned}$$

which is possibly positive only when $(\alpha_{i_j}^m)^{(0)}=(\alpha_{i_0}^{m+n})_j^{(n+1)}$ for all j. We also have that the type structure in the first $m+n+1$ generations of τ_α^{m+n} and τ_α^{m+n+1} are identical with probability 1.

We continue this process and obtain a sequence $\{\tau_\alpha^{m+n}\}_{n\geq 0}$ of almost surely growing random trees. Therefore, the limit $\tau_\alpha\equiv\lim_{n\to\infty}\tau_\alpha^{m+n}$ exists with probability 1. The random element τ_α takes values on $\mathcal{A}^{\bar{d}}$, where $\bar{d}=\max_{\beta\in\mathcal{Q}_m}\big(\max_{0\leq i\leq m}\sigma_\beta^{(i)}\big)$ as defined in Section 2.1, and, for almost every $\omega\in\Omega$, $\tau_\alpha^{m+n}(\omega)$ is the subtree from the root of $\tau_\alpha(\omega)$ to the $(m+n)$th level of $\tau_\alpha(\omega)$, for all $n=0,1,2,\cdots$. We call τ_α the *infinite random spread pattern induced from the random m-pattern* ζ_α *with respect to the random m-spread model* $\mathcal{R}$ *and pattern distribution* f.

So, we can ask the same questions proposed for the topological m-spread model here. Namely,if $\mathcal{O}_a(\tau_\alpha|_{\Delta_r^s(\tau_\alpha)})$ is the occurrences of $a\in\mathcal{A}$ in τ_α from level $r+1$ to level s, then we will ask what happens to the rate

$$s_\alpha(a;\{k_n\}_{n=1}^\infty):=\lim_{n\to\infty}s_\alpha(a;[s_n,s_{n+1}]):=\lim_{n\to\infty}\frac{\mathcal{O}_a(\tau_\alpha|_{\Delta_r^s(\tau_\alpha)})}{|\Delta_r^s(\tau_\alpha)|},$$

where $\{k_n\}_{n=1}^\infty\subseteq\mathbb{N}$ is a sequence of natural numbers with $k_n\to\infty$ as $n\to\infty$ and $s_n=\sum_{i=1}^n k_i$, and

$$s_\alpha(a;k)=\lim_{n\to\infty}s_\alpha(a;[kn,k(n+1)])$$

which is the case when $k_n=kn$ in the above.

3.1.2. *Induced branching processes.* In order to investigate the type structure of the infinite tree τ_α induced by the random m-spread model $\mathcal{R}$, we will construct a multitype branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ on the same probability space $(\Omega,\mathcal{F},\mathbb{P})$ in which each $(m-1)$-pattern is considered as a type. Let $\mathbf{A}=\mathcal{Q}_m^{(0)}$ and note that we have removed some $(m-1)$-patterns from $\mathcal{P}_{m-1}$ to obtain the subset $\mathbf{A}$ as long as none of those patterns has a chance (with probability 0) to be a "parent" or a "child" of any m-pattern in the family initiated by the given initial $(m-1)$-pattern α. Since $\mathcal{P}_{m-1}$ is finite, the set $\mathbf{A}$ is well-defined and finite.

Let $\mathbf{A}=\{\alpha_1,\alpha_2,\cdots,\alpha_{\mathbf{k}}\}$ be the type set for the branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ to be constructed. Let $\tau_\alpha^{m-1}=\alpha=\alpha_{i_0}$ for some $\alpha_{i_0}\in\mathbf{A}$, and let the induced population $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ start with an individual of type $\alpha_{i_0}\in\mathbf{A}$ at time 0, i.e., $\vec{\mathbf{Z}}_0=\vec{e}_{i_0}$, where $\vec{e}_{i_0}$ is the standard unit vector with 1 as its i_0th component and 0 elsewhere. We also write $\{\vec{\mathbf{Z}}_n^{(i_0)}\}_{n\geq 0}$ for $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ when $\vec{\mathbf{Z}}_0=\vec{e}_{i_0}$. After one unit of time, when the type structure of the population in the m-spread model grows from τ_α^{m-1} to τ_α^m, namely the individuals in the mth generation (level) of the population are born. At the same time, in the induced population, we replace the initial $(m-1)$-pattern α_{i_0} with $\sigma^{(1)}_{\tau_\alpha}$, the $(m-1)$-patterns rooted at level 1 in τ^m_α, to obtain the population

$$\left\{(\tau^m_\alpha)^{(1)}_1, (\tau^m_\alpha)^{(1)}_2, \cdots, (\tau^m_\alpha)^{(1)}_{\sigma^{(1)}_{\tau_\alpha}}\right\}$$

at time 1, which are called the *children* of the initial $(m-1)$-pattern in the language of branching process. Note that each $(\tau^m_\alpha)^{(1)}_i$ is a random element taking values in the type set $\mathbf{A}$. Let $\mathbf{Z}^{(i_0)}_{1,i}$ be the number of the $(m-1)$-patterns of type α_i in the population at time 1, $i=1,2,\cdots,\mathbf{k}$, and call the vector $\vec{\mathbf{Z}}^{(i_0)}_1 = (\mathbf{Z}^{(i_0)}_{1,1}, \mathbf{Z}^{(i_0)}_{1,2}, \cdots, \mathbf{Z}^{(i_0)}_{1,\mathbf{k}})$ the population vector at time 1. Then we have that $|\mathbf{Z}^{(\vec{i_0})}_1| = \sigma^{(1)}_{\tau_\alpha}$ and, for any $\vec{r} = (r_1, r_2, \cdots, r_{\mathbf{k}})$,

$$\begin{aligned} & \mathbb{P}^{(i_0)}(\vec{r}) \\ \equiv\ & \mathbb{P}(\vec{\mathbf{Z}}^{(i_0)}_1 = \vec{r}) \text{ which is the probability that an individual of type } \alpha_{i_0} \\ & \text{in the induced process produces } r_j \text{ children of type } \alpha_j,\ j=1,2,\cdots,\mathbf{k}. \\ =\ & \sum_{\substack{\beta\in\mathcal{Q}_m \text{ s.t.} \\ \mathbf{T}(\beta)=\vec{r}}} f(\alpha_{i_0}, \beta) \end{aligned}$$

where $\mathbf{T}(\beta) = (t_1, t_2, \cdots, t_{\mathbf{k}})$ is the vector with t_i as the number of the $(m-1)$-patterns of type α_i among $\left\{\beta^{(1)}_1, \beta^{(1)}_2, \cdots, \beta^{(1)}_{\sigma^{(1)}_\beta}\right\}$, $i=1,2,\cdots,\mathbf{k}$.

Next, after another unit of time, the type structure of the population grows from τ^m_α to τ^{m+1}_α. So, again in the induced population, we replace $\sigma^{(1)}_{\tau_\alpha}$ $(m-1)$-patterns rooted at individuals in the 1st generation with $\sigma^{(2)}_{\tau_\alpha}$ $(m-1)$-patterns

$$\left\{(\tau^{m+1}_\alpha)^{(2)}_1, (\tau^{m+1}_\alpha)^{(2)}_2, \cdots, (\tau^{m+1}_\alpha)^{(2)}_{\sigma^{(2)}_{\tau_\alpha}}\right\}$$

rooted at individuals in the 2nd generation in τ^{m+1}_α to obtain the population vector $\vec{\mathbf{Z}}^{(i_0)}_2 = (\mathbf{Z}^{(i_0)}_{2,1}, \mathbf{Z}^{(i_0)}_{2,2}, \cdots, \mathbf{Z}^{(i_0)}_{2,\mathbf{k}})$ at time 2 and so on. Let $\vec{\mathbf{Z}}^{(i_0)}_n = (\mathbf{Z}^{(i_0)}_{n,1}, \mathbf{Z}^{(i_0)}_{n,2}, \cdots, \mathbf{Z}^{(i_0)}_{n,\mathbf{k}})$ be the population vector at time n, where $\mathbf{Z}^{(i_0)}_{n,i}$ is the number of the $(m-1)$-patterns of type α_i among the population

$$\left\{(\tau^{m+n-1}_\alpha)^{(n)}_1, (\tau^{m+n-1}_\alpha)^{(n)}_2, \cdots, (\tau^{m+n-1}_\alpha)^{(n)}_{\sigma^{(n)}_{\tau_\alpha}}\right\}$$

at time n. Then such a process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is called the *induced process* with the induced type set $\mathbf{A}$ from the random m-spread model $\mathcal{R}$.

From the construction, for almost every realization of the infinite spread pattern τ_α, there is a corresponding realization of the process $|\vec{\mathbf{Z}}_n|$ and each $(m-1)$-pattern rooted at a node in the nth level of τ_α represents an individual in the nth generation of $|\vec{\mathbf{Z}}_n|$. Therefore, we directly obtain the following basic connection between them.

Lemma 11. *If $\alpha = \alpha_{i_0} \in \mathbf{A}$, then, for all $n = 0,1,2,\cdots$, $|\vec{\mathbf{Z}}^{(i_0)}_n| = \sigma^{(n)}_{\tau_\alpha}$ with probability 1.*

If we let

$$\{\vec{\xi}^{(i)}\}^{\mathbf{k}}_{i=1} = \{(\xi^{(i)}_1, \xi^{(i)}_2, \cdots, \xi^{(i)}_{\mathbf{k}})\}^{\mathbf{k}}_{i=1}$$

be the offspring vectors of the induced process $\{\vec{\mathbf{Z}}_n^{(i_0)}\}_{n\geq 0}$, where $\xi_j^{(i)}$ denotes the number of children of type α_j of an individual of type α_i, then

$$\mathbb{P}(\vec{\xi}^{(i)} = \cdot) = \mathbb{P}(\vec{\mathbf{Z}}_1^{(i)} = \cdot) \equiv \mathbb{P}^{(i)}(\cdot)$$

is the offspring distribution.

Moreover, we have that

$$\vec{\mathbf{Z}}_{n+1}^{(i_0)} = \sum_{j=1}^{\sigma_{\tau_\alpha}^{(n)}} \mathbf{T}\big((\tau_\alpha^{m+n})_j^{(n)}\big) = \sum_{i=1}^{\mathbf{k}} \sum_{r=1}^{\mathbf{Z}_{n,i}^{(i_0)}} \vec{\xi}_{n,r}^{(i)}.$$

where $\vec{\xi}_{n,r}^{(i)}$ is the offspring vector of the rth individual of type α_i at time n and has the same distribution as $\vec{\xi}^{(i)}$ for all r and all n. Hence, the induced process $\{\vec{\mathbf{Z}}_n^{(i_0)}\}_{n\geq 0}$ satisfies the characteristics of the branching process and, therefore, is indeed a branching process.

Proposition 12. *The induced process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ from any random m-spread model $\mathcal{R}$ is a multi-type Galton-Watson branching process with the type set $\mathbf{A}$ and the offspring distribution $\{\mathbb{P}^{(i)}(\cdot)\}_{i=1}^{\mathbf{k}}$ as defined above.*

Therefore, according to Section 2.2 in Ban et al (2021), we can construct a random 1-spread model $\mathbf{R} = \big\{\mathbf{p}_{\alpha_i}^{(1)}\big\}_{i=1}^{\mathbf{k}}$ with type set $\mathbf{A} = \{\alpha_i\}_{i=1}^{\mathbf{k}}$ and spread distribution $\{\mathbb{P}^{(i)}(\cdot)\}_{i=1}^{\mathbf{k}}$, for which the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is the underlying branching process. Note that, for each $\alpha \in \mathbf{A}$, each pattern can be written as a tuple

$$\mathbf{p}_\alpha^{(1)} = (\alpha; \alpha_1^{(1)} \cdots \alpha_{\sigma_\alpha^{(1)}}^{(1)}),$$

where $\sigma_\alpha^{(1)}$ is the number of the children of the ancestor α, and if $\alpha = \alpha_i$ for some $i = 1, 2, \cdots, \mathbf{k}$, then $\sigma_\alpha^{(1)} = |\mathbf{Z}_1^{(i)}|$. We also call $\big\{\mathbf{p}_{\alpha_i}^{(1)}\big\}_{i=1}^{\mathbf{k}}$ the *random* 1*-spread model* induced by the random m-spread model $\mathcal{S}$ or *induced* 1*-spread model*.

For $n \in \mathbb{N}$ and $\mathbf{p}_\alpha^{(n)}$, a pattern of the nth generation, the offspring type chart of α in n generations is

$$\Pi(\mathbf{p}_\alpha^{(n)}) = (\alpha_1^{(n)} \cdots \alpha_{\sigma_\alpha^{(n)}}^{(n)}) \in \mathbf{A}^{\sigma_\alpha^{(n)}}$$

and, if $\alpha = \alpha_i$ and $\beta = \alpha_j$, then $\Big|\Pi(\mathbf{p}_\alpha^{(n)})\Big|_\beta = \mathbf{Z}_{n,j}^{(i)}$.

Let $\mathbf{M} = [\mathbf{m}_{ij}]_{\mathbf{k}\times\mathbf{k}}$ be the offspring mean matrix of the induced branching process $\{\mathbf{Z}_n^{(i_0)}\}_{n\geq 0}$, where

$$\mathbf{m}_{ij} = \mathbb{E}(\mathbf{Z}_{1,i}^{(j)}) = \mathbb{E}\Big(\Big|\prod(\mathbf{p}_{\alpha_j}^{(1)})\Big|_{\alpha_i}\Big).$$

and let $\mathbf{M}^{(n)} = [\mathbf{m}_{ij}^{(n)}]_{\mathbf{k}\times\mathbf{k}}$, where

$$\mathbf{m}_{ij}^{(n)} = \mathbb{E}(\mathbf{Z}_{n,i}^{(j)}) = \mathbb{E}\Big(\Big|\prod(\mathbf{p}_{\alpha_j}^{(n)})\Big|_{\alpha_i}\Big).$$

and it is known that $M^{(n)} = M^n$ for all $n = 1, 2, \cdots$.

Since we obtain the type set $\mathbf{A}$ by removing the $(m-1)$-patterns for which it is impossible (with probability 0) to be the "parent" or a "child" of any m-pattern from $\mathcal{P}_{m-1}$, so the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is positive regular. Note that a branching process is said to be *singular* if every individual in the population only produces exactly one child with probability one and, by Lemma 11, we have that the corresponding infinite spread pattern τ_α only has one node in each level with probability one in this case which is not of our interest. Therefore, throughout this paper, we assume that the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ is non-singular. Then, we have the growth rate regarding the induced branching process.

Lemma 13 (Proposition 3.5. and Theorem 3.7. Ban et al (2021))**.** *Let $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ be the induced branching process from the random m-spread model $\mathcal{R}$. Let $\mathbf{M}$ be the offspring mean matrix with the spectral radius $\rho > 1$ and the corresponding left and right eigenvectors $\vec{\mathbf{u}} = (\mathbf{u}_1, \cdots, \mathbf{u}_\mathbf{k})$ and $\vec{\mathbf{v}} = (\mathbf{v}_1, \cdots, \mathbf{v}_\mathbf{k})$ such that $\vec{\mathbf{v}} \cdot \vec{\mathbf{1}} = 1$ and $\vec{\mathbf{v}} \cdot \vec{\mathbf{u}} = 1$. Then,*

(i) *for every $i = 1, 2, \cdots, \mathbf{k}$,*
$$\mathbb{P}(\sigma_{\alpha_i}^{(n)} \to \infty) = 1;$$
(ii) *for all $i, j = 1, 2, \cdots, \mathbf{k}$,*
$$\lim_{n\to\infty} \frac{\mathbf{m}_{ij}^{(n)}}{\rho^n} = \mathbf{v}_i \mathbf{u}_j;$$
(iii) *for every $i = 1, 2, \cdots, \mathbf{k}$, there exists a random variable $\mathbf{W}_i$ such that, for every $j = 1, 2, \cdots, \mathbf{k}$,*
$$\lim_{n\to\infty} \frac{\left|\prod(\mathbf{p}_{\alpha_i}^{(n)})\right|_{\alpha_j}}{\rho^n} = \mathbf{u}_j \mathbf{W}_i \quad \textit{w.p.1};$$
(iv) *for every $i = 1, 2, \cdots, \mathbf{k}$,*
$$s_{\alpha_i}(\alpha_j) = \lim_{n\to\infty} s_{\alpha_i}^{(n)}(\alpha_j) \equiv \lim_{n\to\infty} \frac{\left|\prod(\mathbf{p}_{\alpha_i}^{(n)})\right|_{\alpha_j}}{\sigma_{\alpha_i}^{(n)}} = \mathbf{v}_j \quad \textit{w.p.1.}$$

The above lemma tells that the number of individuals in the nth generation grows geometrically like ρ^n and the proportion of individuals of type type α_i in the nth level eventually will tend to the ith component of the right eigenvector $\vec{\mathbf{v}}$. The next lemma gives us the information about the composition of the types when we look at more than one generation at once. In particular, we consider a sequence $\{k_n\}_{n=1}^\infty$ with $k_n \to \infty$ as $n \to \infty$ and $s_n = \sum_{r=1}^{n} k_n$ and then study the type composition of the individuals from the k_nth generation to the k_{n+1}th generation.

Lemma 14. *Let $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ be the induced branching process from the random m-spread model $\mathcal{R}$. Let $\mathbf{M}$ be the offspring mean matrix with ρ, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ as defined in Lemma 13. Suppose that $\{k_n\}_{n=1}^\infty$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Let $s_n = \sum_{r=1}^{n} k_n$. Then, for any $\alpha_i, \alpha_j \in \mathbf{A}$, we have that*
$$s_{\alpha_i}(\alpha_j; \{k_n\}_{n=1}^\infty) := \lim_{n\to\infty} \frac{\mathcal{O}_{\alpha_j}(\tau_{\alpha_i}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_i})})}{|\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_i})|} = \mathbf{v}_j$$
with probability 1. In particular, if $k_n = k$ for all $n \in \mathbb{N}$, then, we have that
$$s_{\alpha_i}(\alpha_j; k) := \lim_{n\to\infty} \frac{\mathcal{O}_{\alpha_j}(\tau_{\alpha_i}|_{\Delta_{kn+1}^{k(n+1)}(\tau_{\alpha_i})})}{|\Delta_{kn+1}^{k(n+1)}(\tau_{\alpha_i})|} = \mathbf{v}_j$$
with probability 1.

Proof. Let $\mathbf{B}_i = \{\omega \in \Omega : \sigma^{(n)}_{\alpha_i}(\omega) \to \infty\}$ be the event of non-extinction of the induced process $\{\vec{\mathbf{Z}}^{(i)}_n\}_{n=0}^{\infty}$ initiated with $\vec{\mathbf{Z}}^{(i)}_0 = \vec{e}_i$. By Lemma 13 (i) and (iv), we have that $\mathbb{P}(\mathbf{B}_i) = 1$ and, on the event $\mathbf{B}_i$,

$$s_{\alpha_i}(\alpha_j) = \lim_{n\to\infty} s^{(n)}_{\alpha_i}(\alpha_j) \equiv \lim_{n\to\infty} \frac{\left|\prod(\mathbf{p}^{(n)}_{\alpha_i})\right|_{\alpha_j}}{\sigma^{(n)}_{\alpha_i}} = \mathbf{v}_j \quad \text{w.p.1.}$$

So, for any $j = 1, 2, \cdots, \mathbf{k}$, let

$$\mathbf{E}_{i,j} = \left\{\omega \in \mathbf{B_i} : \frac{\left|\prod(\mathbf{p}^{(n)}_{\alpha_i})(\omega)\right|_{\alpha_j}}{\sigma^{(n)}_{\alpha_i}(\omega)} \to \mathbf{v}_j\right\}$$

then we have that $\mathbb{P}(\mathbf{B}_i \setminus \mathbf{E}_{i,j}) = 0$. Hence, $\mathbb{P}(\mathbf{B}_i \setminus \bigcap_{j=1}^{\mathbf{k}} \mathbf{E}_{i,j}) = 0$ and, for each $\omega \in \bigcap_{j=1}^{\mathbf{k}} E_{i,j}$, for every $r = s_n + 1, \cdots, s_{n+1}$, we have that, as $n \to \infty$,

$$\frac{\left|\prod(\mathbf{p}^{(r)}_{\alpha_i})(\omega)\right|_{\alpha_j}}{\sigma^{(r)}_{\alpha_i}(\omega)} \to \mathbf{v}_j \quad \text{and} \quad \sum_{r=s_n+1}^{s_{n+1}} \sigma^{(n)}_{\alpha_i}(\omega) \to \infty.$$

Therefore, by Lemma 2 in Section 2.2, we have that

$$\begin{aligned} & s_{\alpha_i}(\alpha_j; \{k_n\}_{k=1}^{n})(\omega) \\ = & \lim_{n\to\infty} \frac{\mathcal{O}_{\alpha_j}(\tau_{\alpha_i}|_{\Delta^{s_{n+1}}_{s_n}(\tau_{\alpha_i})})}{|\Delta^{s_{n+1}}_{s_n}(\tau_{\alpha_i})|} = \lim_{n\to\infty} \frac{\sum_{r=s_n+1}^{s_{n+1}} \left|\prod(\mathbf{p}^{(r)}_{\alpha_i})(\omega)\right|_{\alpha_j}}{\sum_{r=s_n+1}^{s_{n+1}} \sigma^{(r)}_{\alpha_i}(\omega)} = \mathbf{v}_j \end{aligned}$$

which completes the proof of the first part of the lemma and the similar lines can be adopted to show the second part.

□

Lemma 14 tells us that if we look at the partial type structure from generation s_n+1 to generation s_{n+1} in the branching tree $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$, the rate of the occurrences of a certain $(m-1)$-pattern α_j converges to a deterministic value $\mathbf{v}_j$ as $n \to \infty$. This idea will give us the answer to the original questions of interest regarding the infinite spread pattern τ_α induced from the random m-pattern ζ_α mentioned previously.

3.2. **Main results.**

3.2.1. *The spread rates of a type (symbol) and a pattern.* Recall that τ_α is the infinite spread pattern induced from the random m-pattern ζ_α with respect to the random m-spread model and the type set $\mathcal{A} = \{a_i\}_{i=1}^{k}$ and $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$ is its corresponding induced branching process with type set $\mathbf{A} = \{\alpha_i\}_{i=1}^{\mathbf{k}}$.

For each $a \in \mathcal{A}$, we define

$$\theta(a) = \{\beta \in \mathbf{A} : \epsilon(\beta) = a\}$$

where $\epsilon(\beta)$ is the root of the pattern β.

Theorem 15. *Let τ_α be the infinite spread pattern with the type set $\mathcal{A} = \{a_i\}_{i=1}^{k}$ and $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$ be its corresponding induced branching process with type set $\mathbf{A} = \{\alpha_i\}_{i=1}^{\mathbf{k}}$. Let $\mathbf{M}$ be the offspring mean matrix of $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$ with ρ, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ as* *defined in Lemma 13. Suppose that $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive integers such that $k_n \to \infty$ as $n \to \infty$. Let $s_n = \sum_{r=1}^{n} k_n$. Then, for any $a_j \in \mathcal{A}$, we have that*

$$s_\alpha(a_j;\{k_n\}_{n=1}^{\infty}) = \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,\mathbf{k}\ s.t.\\ \alpha_i\in\theta(a_j)}} \mathbf{v}_i$$

with probability 1. In particular, if $k_n = kn$ for all $n \in \mathbb{N}$, then, we have that

$$s_\alpha(a_j;k) = \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)})}{\left|\Delta_{kn}^{k(n+1)}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,\mathbf{k}\ s.t.\\ \alpha_i\in\theta(a_j)}} \mathbf{v}_i$$

with probability 1.

Proof. Let $\alpha = \alpha_{i_0} \in \mathbf{A}$ and let $\mathbf{B}_{i_0}$ and $\mathbf{E}_{i_0} = \bigcap_{j=1}^{\mathbf{k}} \mathbf{E}_{i_0,j}$ be defined as in the proof of Lemma 14. Then $\mathbb{P}(\mathbf{B}_{i_0}) = 1$, $\mathbb{P}(\mathbf{B}_{i_0} \setminus \mathbf{E}_{i_0}) = 0$, and for every $\omega \in \mathbf{E}_{i_0}$, as $n \to \infty$,

$$\frac{\sum_{r=s_n+1}^{s_{n+1}} \left|\prod(\mathbf{p}_{\alpha_{i_0}}^{(r)})(\omega)\right|_{\alpha_j}}{\sum_{r=s_n+1}^{s_{n+1}} \sigma_{\alpha_{i_0}}^{(r)}(\omega)} \to \mathbf{v}_j$$

for all $j = 1, 2, \cdots, \mathbf{k}$. Note that, by the construction of the induced process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ from the infinite spread pattern τ_α, we have that for each $\omega \in \mathbf{E}_{i_0}$,

$$\left|\Delta_{s_n}^{s_{n+1}}(\tau_\alpha(\omega))\right| = \sum_{r=s_n+1}^{s_{n+1}} \sigma_{\tau_\alpha(\omega)}^{r} = \sum_{r=s_n+1}^{s_{n+1}} \left|\vec{\mathbf{Z}}_r^{(i_0)}\right| = \sum_{r=s_n+1}^{s_{n+1}} \sigma_{\alpha_{i_0}}^{(r)}$$

and, since $\mathbf{Z}_{r,i}^{(i_0)}$ is the number of the $(m-1)$-pattern of type $\alpha_i \in \mathbf{A}$ among

$$\left\{(\tau_\alpha^{m+r-1})_1^{(n)}, (\tau_\alpha^{m+r-1})_2^{(n)}, \cdots, (\tau_\alpha^{m+r-1})_{\sigma_{\tau_\alpha}^{(r)}}^{(n)}\right\},$$

it implies that

$$\begin{aligned}
& \mathcal{O}_{a_j}(\tau_\alpha(\omega)|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha(\omega))}) \\
= & \sum_{r=s_n+1}^{s_{n+1}} (\text{ the number of occurrences of } a_j \text{ in the } r\text{th level of } \tau_\alpha(\omega)) \\
= & \sum_{r=s_n+1}^{s_{n+1}} (\text{ the number of occurrences of } a_j \text{ in the } r\text{th level of } \tau_\alpha^{m+r-1}(\omega)) \\
= & \sum_{r=s_n+1}^{s_{n+1}} (\text{ the number of the } (m-1)\text{-pattern with the root of type } a_j \\
& \quad \text{at the } r\text{th level of } \tau_\alpha^{m+r-1}(\omega)) \\
= & \sum_{r=s_n+1}^{s_{n+1}} \left(\sum_{\substack{i=1,2,\cdots,\mathbf{k}\text{ s.t.}\\ \alpha_i\in\theta(a_j)}} \mathbf{Z}_{r,i}^{(i_0)}(\omega) \right) \\
= & \sum_{r=s_n+1}^{s_{n+1}} \left(\sum_{\substack{i=1,2,\cdots,\mathbf{k}\text{ s.t.}\\ \alpha_i\in\theta(a_j)}} \left|\prod(\mathbf{p}_{\alpha_{i_0}}^{(r)}(\omega))\right|_{\alpha_i} \right).
\end{aligned}$$

Therefore, by Lemma 14, for each $\omega \in \mathbf{E}_{i_0}$ with $\mathbb{P}(\mathbf{B}_{i_0} \setminus \mathbf{E}_{i_0}) = 0$,

$$\begin{aligned}
& s_\alpha(a_j; \{k_n\}_{n=1}^\infty)(\omega) \\
= & \lim_{n\to\infty} \frac{\mathcal{O}_{a_j}(\tau_\alpha(\omega)|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})}{\left|\Delta_{s_n}^{s_{n+1}}(\tau_\alpha(\omega))\right|} \\
= & \lim_{n\to\infty} \frac{\sum_{r=s_n+1}^{s_{n+1}} \left(\sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i \in \theta(a_j)}} \left| \prod(\mathbf{p}_{\alpha_{i_0}}^{(r)}(\omega)) \right|_{\alpha_i} \right)}{\sum_{r=s_n+1}^{s_{n+1}} \sigma_{\alpha_{i_0}}^{(r)}(\omega)} \\
= & \sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i \in \theta(a_j)}} \lim_{n\to\infty} \frac{\sum_{r=s_n+1}^{s_{n+1}} \left(\left| \prod(\mathbf{p}_{\alpha_{i_0}}^{(r)}(\omega)) \right|_{\alpha_i} \right)}{\sum_{r=s_n+1}^{s_{n+1}} \sigma_{\alpha_{i_0}}^{(r)}(\omega)} \\
= & \sum_{\substack{i=1,2,\cdots,\mathbf{k} \text{ s.t.} \\ \alpha_i \in \theta(a_j)}} \mathbf{v}_i
\end{aligned}$$

and, since $\mathbb{P}(\mathbf{B}_{i_0}) = 1$, the proof is complete.

□

The following example gives us the main ideas of Theorem 15:

Example 16. *Consider there are two types, a and b, that is, the type set is $\mathcal{A} = \{a, b\}$. Let $m = 2$ and it means that the spread depends on the type structure in the past 2 levels (i.e., generations).*

Let $\mathcal{Q}_2^{(0)} = \{\alpha_1, \cdots, \alpha_5\}$ and $\mathcal{Q}_2 = \{\alpha_1^2, \cdots, \alpha_{10}^2\}$be defined as illustrated in Figure 7(a) and in the bottom of Figure 7(c).

Let $f : \mathcal{Q}_2^{(0)} \times \mathcal{Q}_2 \to [0, 1]$ be a pattern distribution function such that

$$f(\alpha_1, \alpha_i^2) = \begin{cases} \frac{2}{3}, & \text{if } i = 1; \\ \frac{1}{3}, & \text{if } i = 2; \\ 0, & \text{otherwise;} \end{cases} \qquad f(\alpha_2, \alpha_i^2) = \begin{cases} \frac{3}{4}, & \text{if } i = 3; \\ \frac{1}{4}, & \text{if } i = 4; \\ 0, & \text{otherwise;} \end{cases}$$

$$f(\alpha_3, \alpha_i^2) = \begin{cases} \frac{1}{2}, & \text{if } i = 5; \\ \frac{1}{2}, & \text{if } i = 6; \\ 0, & \text{otherwise;} \end{cases} \qquad f(\alpha_4, \alpha_i^2) = \begin{cases} \frac{1}{3}, & \text{if } i = 7; \\ \frac{2}{3}, & \text{if } i = 8; \\ 0, & \text{otherwise;} \end{cases}$$

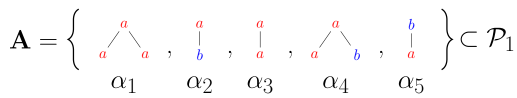
and

$$f(\alpha_5, \alpha_i^2) = \begin{cases} \frac{1}{2}, & \text{if } i = 9; \\ \frac{1}{2}, & \text{if } i = 10; \\ 0, & \text{otherwise.} \end{cases}$$

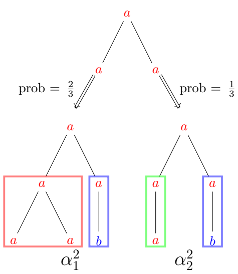
Note that each 2-pattern in $\mathcal{Q}_2$ can be represented s follows:

$$\begin{array}{llll}
\alpha_1^2 = (\alpha_1; \alpha_1, \alpha_2); & \alpha_2^2 = (\alpha_1; \alpha_2, \alpha_2); & \alpha_3^2 = (\alpha_2; \alpha_4, \alpha_5); & \alpha_4^2 = (\alpha_2; \alpha_5); \\
\alpha_5^2 = (\alpha_3; \alpha_1); & \alpha_6^2 = (\alpha_3; \alpha_2); & \alpha_7^2 = (\alpha_4; \alpha_3, \alpha_4); & \alpha_8^2 = (\alpha_4; \alpha_3, \alpha_5); \\
\alpha_9^2 = (\alpha_5; \alpha_3); & \alpha_{10}^2 = (\alpha_5; \alpha_1) & &
\end{array}$$

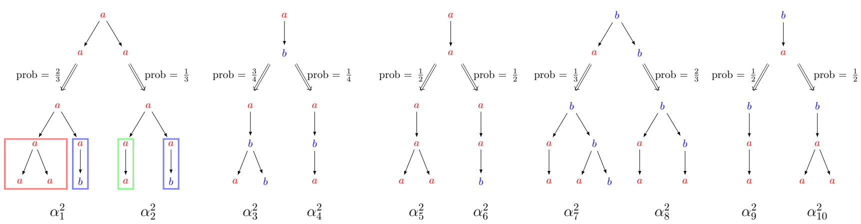
Moreover, Figure 7(b) gives an idea how α_1 grows into 2-patterns with the corresponding probabilities and Figure 7(c) is an illustration of the pattern distribution f. So, in this case, if we let $\mathcal{R} = \{\zeta_{\alpha_1}, \zeta_{\alpha_2}, \zeta_{\alpha_3}, \zeta_{\alpha_4}, \zeta_{\alpha_5}\}$, then $\mathcal{R}$ is a random 2-spread



(a) The type set $\mathbf{A}$



(b) The spread probability $f(\alpha_1, \cdot)$ from α_1



(c) The spread distribution

FIGURE 7. Illustration of Example 16

model generated by the pattern distribution f, where ζ_α is a random 2-pattern and the distribution of ζ_α is given by

$$\mathbb{P}(\zeta_\alpha = \beta) = f(\alpha, \beta) = \begin{cases} f(\alpha_i, \alpha_j^2), & \text{if } \alpha = \alpha_i \text{ and } j = 2i-1, 2i \\ 0, & \text{otherwise.} \end{cases}$$

Now, consider the infinite spread pattern τ_α with the initial 1-pattern $\alpha = \alpha_1$ induced from the random 2-pattern ζ_{α_1} with respect to the random spread model $\mathcal{R}$. Let $\mathbf{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ be the type set for the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$. From Figure 7(b) or Figure 7(c), we can see that, in the induced branching process, one individual of type α_1 can produce one child of type α_1 and one child of type α_3 with probability $\frac{2}{3}$ or can produce one child of type α_2 and one child of type α_3 with probability $\frac{1}{3}$, so its offspring means are

$$\begin{aligned} m_{11} &= \mathbb{E}(\mathbf{Z}_{11}|\vec{\mathbf{Z}}_0 = \vec{e}_1) = 1 \cdot \tfrac{2}{3} = \tfrac{2}{3}; \\ m_{21} &= \mathbb{E}(\mathbf{Z}_{12}|\vec{\mathbf{Z}}_0 = \vec{e}_1) = 1 \cdot \tfrac{2}{3} + 1 \cdot \tfrac{1}{3} = 1; \\ m_{31} &= \mathbb{E}(\mathbf{Z}_{13}|\vec{\mathbf{Z}}_0 = \vec{e}_1) = 1 \cdot \tfrac{1}{3} = \tfrac{1}{3}; \\ m_{41} &= \mathbb{E}(\mathbf{Z}_{14}|\vec{\mathbf{Z}}_0 = \vec{e}_1) = 0; \\ m_{41} &= \mathbb{E}(\mathbf{Z}_{14}|\vec{\mathbf{Z}}_0 = \vec{e}_1) = 0. \end{aligned}$$

Similarly, we can compute all the offspring means of individuals of different types:

$$\begin{aligned}
m_{42} &= \mathbb{E}(\mathbf{Z}_{14}|\vec{\mathbf{Z}}_0 = \vec{e}_2) = 1 \cdot \tfrac{3}{4} = \tfrac{3}{4};\\
m_{52} &= \mathbb{E}(\mathbf{Z}_{15}|\vec{\mathbf{Z}}_0 = \vec{e}_2) = 1 \cdot \tfrac{1}{4} = \tfrac{1}{4};\\
m_{13} &= \mathbb{E}(\mathbf{Z}_{11}|\vec{\mathbf{Z}}_0 = \vec{e}_3) = \tfrac{1}{2};\\
m_{23} &= \mathbb{E}(\mathbf{Z}_{12}|\vec{\mathbf{Z}}_0 = \vec{e}_3) = \tfrac{1}{2};\\
m_{34} &= \mathbb{E}(\mathbf{Z}_{13}|\vec{\mathbf{Z}}_0 = \vec{e}_4) = 1;\\
m_{44} &= \mathbb{E}(\mathbf{Z}_{14}|\vec{\mathbf{Z}}_0 = \vec{e}_4) = \tfrac{1}{3};\\
m_{54} &= \mathbb{E}(\mathbf{Z}_{15}|\vec{\mathbf{Z}}_0 = \vec{e}_4) = \tfrac{2}{3};\\
m_{15} &= \mathbb{E}(\mathbf{Z}_{11}|\vec{\mathbf{Z}}_0 = \vec{e}_5) = \tfrac{1}{2};\\
m_{35} &= \mathbb{E}(\mathbf{Z}_{13}|\vec{\mathbf{Z}}_0 = \vec{e}_5) = \tfrac{1}{2}
\end{aligned}$$

and m_{ij}*=0, otherwise. Thus, we obtain the offspring mean matrix of the induced branching process* $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$*:*

$$\mathbf{M} = \begin{bmatrix} \frac{2}{3} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 1 & \frac{1}{2} \\ 0 & \frac{3}{4} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{4} & 0 & \frac{2}{3} & 0 \end{bmatrix}.$$

Moreover, the spectral radius of $\mathbf{M}$ *is* $\rho \approx 1.4$ *and its corresponding normalized right eigenvector* $\vec{\mathbf{v}} \approx (0.23, 0.25, 0.22, 0.17, 0.13)$*. Note that* $\theta(a) = \{\alpha_1, \alpha_2, \alpha_3\}$*, which is the set of patterns rooted at an individual of type a, and* $\theta(b) = \{\alpha_4, \alpha_5\}$*. So, if we take* $k = 1$ *in Theorem 15, then we can obtain the spread rate of type a:*

$$s_\alpha(a; 1) = v_1 + v_2 + v_3 = 0.23 + 0.25 + 0.22 = 0.7$$

and the spread rate of type b is

$$s_\alpha(b; 1) = v_4 + v_5 = 0.17 + 0.13 = 0.3.$$

That means, in the long run, the proportion of individuals of type a and type b are about 70% *and* 30%*, respectively, among the population.*

For any r-pattern $\alpha_j^r \in \mathcal{P}_r$, $1 \leq r \leq m-1$, let

$$\theta_r(\alpha_j^r) = \{\beta \in \mathbf{A} : \beta|_{\Delta_0^r(\beta)} = \alpha_j^r\}$$

and let $\mathcal{O}_{\alpha_j^r}(\tau_\alpha|_{\Delta_s^t(\tau_\alpha)})$ be the occurrences of the pattern α_j^r in $\Delta_s^t(\tau_\alpha)$, where $t-s \geq r$. Then the following theorem gives the spread rate of the patter α_j^r in the infinite spread pattern τ_α:

Theorem 17. *Let* τ_α *be the infinite spread pattern with the type set* $\mathcal{A} = \{a_i\}_{i=1}^k$ *and* $\{\vec{\mathbf{Z}}_n\}_{n=0}^\infty$ *be its corresponding induced branching process with type set* $\mathbf{A} = \{\alpha_i\}_{i=1}^{\mathbf{k}}$*. Let* $\mathbf{M}$ *be the offspring mean matrix of* $\{\vec{\mathbf{Z}}_n\}_{n=0}^\infty$ *with* ρ*,* $\vec{\mathbf{u}}$ *and* $\vec{\mathbf{v}}$ *as defined in Lemma 13. Let* $1 \leq r \leq m-1$ *be any fixed positive integer. Suppose that* $\{k_n\}_{n=1}^\infty$ *is a sequence of positive integers such that* $k_n \geq r$ *for each* n *and* $k_n \to \infty$ *as* $n \to \infty$*. Let* $s_n = \sum_{r=1}^{n} k_n$*. Then, for any* $\alpha_j^r \in \mathcal{P}_r$*, we have that*

$$s_\alpha(\alpha_j^r; \{k_n\}_{n=1}^\infty) = \lim_{n\to\infty} \frac{\mathcal{O}_{\alpha_j^r}(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_\alpha)\right|} = \sum_{\substack{i=1,2,\cdots,l \ s.t. \\ \alpha_i \in \theta_r(\alpha_j^r)}} \mathbf{v}_i$$

with probability 1.

Proof. Since, for each $\omega \in \mathbf{E}_{i_0}$ defined previously,

$$\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_\alpha(\omega))\right| = \sum_{t=s_n+1}^{s_{n+1}-r} \sigma^t_{\tau_\alpha(\omega)} = \sum_{r=s_n+1}^{s_{n+1}-r} \left|\vec{\mathbf{Z}}_t^{(i_0)}\right| = \sum_{t=s_n+1}^{s_{n+1}-r} \sigma^{(t)}_{\alpha_{i_0}}$$

and

$$\begin{aligned}
&\mathcal{O}_{\alpha_j^r}(\tau_\alpha(\omega)|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha(\omega))}) \\
= &\sum_{t=s_n+1}^{s_{n+1}-r} (\text{ the number of occurrences of the } r\text{-pattern } \alpha_j^r \text{ rooted in} \\
&\qquad \text{the } t\text{th level of } \tau_\alpha(\omega)) \\
= &\sum_{t=s_n+1}^{s_{n+1}-r} (\text{ the number of occurrences of the } r\text{-pattern } \alpha_j^r \text{ rooted in} \\
&\qquad \text{the } t\text{th level of } \tau_\alpha^{m+t-1}(\omega)) \\
= &\sum_{t=s_n+1}^{s_{n+1}-r} (\text{ the number of the } (m-1)\text{-pattern } \beta \text{ with the root at} \\
&\qquad \text{the } r\text{th level of } \tau_\alpha^{m+t-1}(\omega) \text{ such that } \beta|_{\Delta_0^r(\beta)} = \alpha_j^r) \\
= &\sum_{t=s_n+1}^{s_{n+1}-r} \left(\sum_{\substack{i=1,2,\cdots,l \text{ s.t.} \\ \alpha_i \in \theta_r(\alpha_j^r)}} \mathbf{Z}_{t,i}^{(i_0)}(\omega) \right) \\
= &\sum_{t=s_n+1}^{s_{n+1}-r} \left(\sum_{\substack{i=1,2,\cdots,l \text{ s.t.} \\ \alpha_i \in \theta_r(\alpha_j^r)}} \left| \prod (\mathbf{p}^{(t)}_{\alpha_{i_0}}(\omega)) \right|_{\alpha_i} \right),
\end{aligned}$$

the result can be shown by the similar analogy as in the proof of Theorem 15. □

3.2.2. *Comparison of two random models.* Let $\mathcal{R}$ and $\mathcal{R}'$ be two random m-spread models with the same type set $\mathcal{A}$ and let τ_α and τ'_α be the infinite spread patterns $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ and $\{\vec{\mathbf{Z}}'_n\}_{n\geq 0}$ be the corresponding branching processes with the type set $\mathbf{A}$ induced by $\mathcal{R}$ and $\mathcal{R}'$,respectively. Let $\mathbf{M} = [\mathbf{m}_{i,j}]_{\mathbf{k}\times\mathbf{k}}$ and $\mathbf{M}' = [\mathbf{m}'_{i,j}]_{\mathbf{k}\times\mathbf{k}}$ be the offspring mean matrices for $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ and $\{\vec{\mathbf{Z}}'_n\}_{n\geq 0}$, respectively. Define $\mathbf{d} = \min_{\mathbf{a}_i\in\mathbf{A}} \mathbb{E}|\mathbf{Z}_1^{(i)}|$ and $\mathbf{D} = \max_{\mathbf{a}_i\in\mathbf{A}} \mathbb{E}|\mathbf{Z}_1^{(i)}|$ for $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$. Respectively, we define $\mathbf{d}'$ and $\mathbf{D}'$ for $\{\vec{\mathbf{Z}}'_n\}_{n\geq 0}$.

Theorem 18. *Suppose that $\mathcal{R}$ and $\mathcal{R}'$ are two random m-spread models with same type set $\mathcal{A}$ and $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$ and $\{\vec{\mathbf{Z}}'_n\}_{n\geq 0}$ are the corresponding induced branching processes with the type set $\mathbf{A}$. Let τ_α, τ'_α, $\mathbf{d}$, $\mathbf{D}$, $\mathbf{d}'$ and $\mathbf{D}'$ be defined as above. Then*

(i) *If $\mathbf{D}' < \mathbf{d}$, then, for any $a_j \in \mathcal{A}$,*

$$\lim_{n\to\infty} \frac{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau'_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau'_\alpha)})\big)}{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})\big)} = 0,$$

where $\{s_n\}_{n=1}^\infty$ is as defined before. In addition, if we let $s_n = kn$ for $k \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau'_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau'_\alpha)})\big)}{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)})\big)} = 0.$$

(ii) *If* $\mathbf{m}'_{i,j} \leq \mathbf{m}_{i,j}$ *for all* $1 \leq i, j \leq \mathbf{k}$ *and there exists a pair* (i_0, j_0) *such that* $\mathbf{m}'_{i_0,j_0} < \mathbf{m}_{i_0,j_0}$*, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau'_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau'_\alpha)})\big)}{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{s_n}^{s_{n+1}}(\tau_\alpha)})\big)} = 0,$$

where $\{s_n\}_{n=1}^{\infty}$ *is defined as before. In addition, if we let* $s_n = kn$ *for* $k \in \mathbb{N}$*, then*

$$\lim_{n\to\infty} \frac{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau'_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau'_\alpha)})\big)}{\mathbb{E}\big(\mathcal{O}_{a_j}(\tau_\alpha|_{\Delta_{kn}^{k(n+1)}(\tau_\alpha)})\big)} = 0.$$

Proof. For the induced branching process $\{\vec{\mathbf{Z}}_n\}_{n\geq 0}$, since

$$\mathbb{E}\Big(\big|\prod(\mathbf{p}_{\alpha_i}^{(n)})\big|_{\alpha_j}\Big) = \mathbf{m}_{j,i}^{(n)} \qquad \text{for } i, j = 1, 2, \cdots, \mathbf{k},$$

we have

$$\sum_{i=1}^{\mathbf{k}} \mathbb{E}\Big(\big|\prod(\mathbf{p}_{\alpha_i}^{(n)})\big|_{\alpha_j}\Big) = \sum_{i=1}^{\mathbf{k}} \mathbf{m}_{j,i}^{(n)} \qquad \text{for } j = 1, 2, \cdots, \mathbf{k}.$$

Similarly, for the other induced branching process $\{\vec{\mathbf{Z}}'_n\}_{n\geq 0}$, we have

$$\sum_{i=1}^{\mathbf{k}} \mathbb{E}\Big(\big|\prod(\mathbf{p}_{\alpha_i}^{(n)'})\big|_{\alpha_j}\Big) = \sum_{i=1}^{\mathbf{k}} \mathbf{m}_{j,i}^{'(n)} \qquad \text{for } j = 1, 2, \cdots, \mathbf{k}.$$

Therefore, after taking the expectation on the random variables, the proofs in the deterministic case can be adopted to prove the results in the random case.

□

Remark 19. *Theorem 18 provides a strategy to compare two random m-spread models. Note that*

$$\mathbb{E}|\mathbf{Z}_1^{(i)}| = \sum_{j=1}^{\mathbf{k}} \mathbb{E}\mathbf{Z}_{1j}^{(i)} = \sum_{j=1}^{\mathbf{k}} \mathbf{m}_{ji}$$

is the sum of all the entries in the ith column of the mean matrix $\mathbf{M}$*. Thus, the result in (i). above tells us that, by comparing the minimum and maximum column sums or the corresponding entries of the mean matrices induced by two m-spread models, we can determine which model will eventually have a greater number of occurrences of a given type. This strategy can be used for decision-making about the epidemic disease prevention and control.*

4. Connection between topological models and random models

A random m-spread model with a special m-pattern distribution can be viewed as a topological m-spread model.

Proposition 20. *Let* $\mathcal{R} = \{\zeta_{\alpha_i}\}_{i=1}^{l}$ *be a random m-spread model with pattern distribution f on* $\mathcal{Q}_m^{(0)} \times \mathcal{Q}_m$*. If, for each* $\alpha_i \in \mathcal{Q}_m^{(0)}$*, there exits exactly one* $\beta_i \in \mathcal{Q}_m$ *such that* $(\beta_i)^{(0)} = \alpha_i$ *and* $f(\alpha_i, \beta_i) = 1$*, then* $\mathcal{R}$ *is a topological m-spread model with probability 1.*

Proof. Since, for each $\alpha_i \in \mathcal{Q}_m^{(0)}$, there exists a $\beta_i \in \mathcal{Q}_m$ such that $f(\alpha_i, \beta_i) = 1 > 0$, if, for each i, let $F_i = \{\omega \in \Omega : \zeta_{\alpha_i}(\omega) \neq \beta_i\}$, then

$$\mathbb{P}(F_i) = 1 - \mathbb{P}(\zeta_{\alpha_i} = \beta_i) = 1 - f(\alpha_i, \beta_i) = 0.$$

Hence, if we let $F = \cup_{i=1}^{l} F_i$, then $\mathbb{P}(F) = 0$ and, for every $\omega \in \Omega \setminus F$, we have that $\zeta_{\alpha_i}(\omega) = \beta_i$ and

$$\mathcal{R}(\omega) = \{\zeta_{\alpha_i}(\omega), \cdots, \zeta_{\alpha_l}(\omega)\} = \{\beta_1, \cdots, \beta_l\}.$$

Now, we want to claim that $\mathcal{R}(\omega) = \{\beta_1, \cdots, \beta_l\}$ is a topological m-spread model for all $\omega \in \Omega \setminus F$. So, for any fixed $\omega \in \Omega \setminus F$ and for any $\beta_i = \zeta_{\alpha_i}(\omega) \in \mathcal{R}(\omega)$, since $\zeta_{\alpha_i} \in \mathcal{R}$ and $f(\alpha_i, \beta_i) = 1 > 0$, we know that $\alpha_i \in \mathcal{Q}_m^{(0)}$ and $\beta_i \in \mathcal{Q}_m$. Hence, $(\beta_i)_j^{(1)} \in \mathcal{Q}_m^{(0)}$ for all j which implies that

$$\zeta_{(\beta_i)_j^{(1)}} \in \mathcal{R}, \text{ for all } j = 1, 2, \cdots, \sigma_{\beta_i}^{(1)}.$$

and thus, for all $j = 1, 2, \cdots, \sigma_{\beta_i}^{(1)}$,

$$\zeta_{(\beta_i)_j^{(1)}} = \zeta_{\alpha_{j'}}, \text{ for some } \zeta_{\alpha_{j'}} \in \mathcal{R}.$$

Therefore, there exists exactly one $\beta_{j'} \in \mathcal{Q}_m$ such that $f(\alpha_{j'}, \beta_{j'}) = 1 > 0$ and $\zeta_{\alpha_{j'}}(\omega) = \beta_{j'}$. This implies that, for any $\beta_i \in \mathcal{R}(\omega)$ and any $j = 1, 2, \cdots, \sigma_{\beta_i}^{(1)}$, there exists exactly one $\beta_{j'} \in \mathcal{R}(\omega)$ and $(\beta_{j'})^{(0)} = \alpha_{j'} = (\beta_i)_j^{(1)}$. So, $\mathcal{R}(\omega)$ is a topological m-spread model for all $\omega \in \Omega \setminus F$. Note that $\mathbb{P}(F) = 0$ and hence $\mathcal{R}$ is a topological m-spread model with probability 1. □

Usually, when some strategies are applied to control the spread of a disease, the spread pattern will change gradually over time, and therefore, we often can see a mixed pattern or a transition phase. For this reason, we want to develop a transition model to describe this phenomenon.

Given two topological m-spread models $\mathcal{S}$ and $\mathcal{S}'$ over the same type set $\mathcal{A}$ and with the same parent set $\mathcal{S}^{(0)} = \mathcal{S}'^{(0)} = \{\alpha_i\}_{i=1}^{l} \subseteq \mathcal{P}_{m-1}$, where $l = |S^{(0)}|$. Note that, for every $\alpha \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$, there exist exactly one $\beta \in \mathcal{S}$ and one $\beta' \in \mathcal{S}'$ such that $\beta^{(0)} = \beta'^{(0)} = \alpha$.

A random m-spread model $\mathcal{R}$ with pattern distribution f is called a *transition model* from $\mathcal{S}$ to $\mathcal{S}'$, if, for any $\alpha_i \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$, $\beta \in \mathcal{S}$, $\beta' \in \mathcal{S}'$ with $\alpha_i = \beta^{(0)} = \beta'^{(0)}$, the pattern distribution f satisfies the following:

(i) if $\beta = \beta'$, then $f(\alpha_i, \beta) = f(\alpha_i, \beta') = 1$ and let $x_i = 1$;
(ii) if $\beta \neq \beta'$, then there exists a number $0 \leq x_i \leq 1$ such that

$$f(\alpha_i, \beta) = 1 - x_i \qquad \text{and } f(\alpha_i, \beta') = x_i.$$

In this case, let $\vec{x} = (x_1, \cdots, x_l)$ and we write $\mathcal{S}_R(\vec{x})$ for $\mathcal{R}$.

Proposition 21. *Let $\mathcal{S}_R(\vec{x})$ be a random m-spread model with pattern distribution f a transition model from the topological m-spread model $\mathcal{S}$ to the topological m-spread model $\mathcal{S}'$. Suppose that, for every $\alpha \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$, $\beta \in \mathcal{S}$, $\beta' \in \mathcal{S}'$ with $\alpha_i = \beta^{(0)} = \beta'^{(0)}$, we have $\beta \neq \beta'$. Then*

(i) *If $\vec{x} = \vec{1}$, then $\mathcal{S}_R(\vec{1}) = \mathcal{S}'$ with probability 1.*
(ii) *If $\vec{x} = \vec{0}$, then $\mathcal{S}_R(\vec{0}) = \mathcal{S}$ with probability 1.*

Proof. The proof is straightforward if we adopt similar lines as those in the proof of the previous proposition. □

Remark 22. *If there exists $i = 1, \cdots, l$ such that $\alpha_i \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$ and $\beta = \beta'$ for $\beta \in \mathcal{S}$, $\beta' \in \mathcal{S}'$ with $\alpha_i = \beta^{(0)} = \beta'^{(0)}$, then (ii) in Proposition 21 can be modified* *into that* $\mathcal{S}_R(\vec{x}^*) = \mathcal{S}$ *with probability 1 where* $\vec{x}^* = (x_1^*, \cdots, x_l^*)$ *with* $x_i^* = 1$ *and* $x_j^* = 0$*, elsewhere.*

Since the transition spread model $\mathcal{S}_R(\vec{x})$ is a random m-spread model, it induces a branching process $\{\vec{\mathbf{Z}}_n\}_{n=0}^{\infty}$ with type set $\mathbf{A} = \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$. By Ban et al (2021) (see also Section 3.1.2 in this paper), there is a corresponding random 1-spread model $\mathbf{R} = \{\mathbf{p}_{\alpha_i}^{(1)}\}_{i=1}^{l}$ to this multi-type branching process.

Now, letting $\mathbf{S}$ and $\mathbf{S}'$ be the corresponding induced 1-spread model of $\mathcal{S}$ and $\mathcal{S}'$ over $\mathbf{A}$, respectively, we have the following proposition.

Proposition 23. *If* $\mathcal{R}$ *is a transition* m*-spread model from* $\mathcal{S}$ *to* $\mathcal{S}'$*, then* $\mathbf{R}$ *is a transition* 1*-spread model from* $\mathbf{S}$ *to* $\mathbf{S}'$*. More precisely, if* $\mathcal{R} = \mathcal{S}_R(\vec{x})$ *for some* $\vec{0} \leq \vec{x} \leq \vec{1}$*, then* $\mathbf{R} = \mathbf{S}_R(\vec{x})$ *with probability 1.*

Proof. Without loss of generality, we assume that, for each $\alpha_i \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$, $\beta \in \mathcal{S}$ and $\beta' \in \mathcal{S}$ with $\beta^{(0)} = \beta'^{(0)} = \alpha_i$, $\beta \neq \beta'$. Then, since $\mathcal{R}$ is a transition m-spread model from $\mathcal{S}$ to $\mathcal{S}'$, there exists a vector $\vec{0} \leq \vec{x} \leq \vec{1}$ such that $\mathcal{R} = \mathcal{S}_R(\vec{x})$.

Recall that in Section 3.1.2, for $\beta \in \mathcal{Q}_m$, $\mathbf{T}(\beta) = (t_1, t_2, \cdots, t_l)$ is the vector with t_k as the number of the $(m-1)$-patterns of type α_k among $\left\{\beta_1^{(1)}, \beta_2^{(1)}, \cdots, \beta_{\sigma_\beta^{(1)}}^{(1)}\right\}$, $k = 1, 2, \cdots, l$. So, for each $\alpha_i \in \mathcal{S}^{(0)} = \mathcal{S}'^{(0)}$, for any $\beta \in \mathcal{S}$ and $\beta' \in \mathcal{S}$ with $\beta^{(0)} = \beta'^{(0)} = \alpha_i$, we have that

$$T(\beta) = \left(\left|\prod(\beta)\right|_{\alpha_1}, \cdots, \left|\prod(\beta)\right|_{\alpha_l}\right)$$

and

$$T(\beta') = \left(\left|\prod(\beta')\right|_{\alpha_1}, \cdots, \left|\prod(\beta')\right|_{\alpha_l}\right).$$

Then

$$\begin{aligned} & \mathbb{P}^{(i)}\left(\left|\prod(\beta)\right|_{\alpha_1}, \cdots, \left|\prod(\beta)\right|_{\alpha_l}\right) = \mathbb{P}^{(i)}(T(\beta)) = \mathbb{P}(\vec{\mathbf{Z}}_1^{(i)} = T(\beta)) \\ = \quad & f(\alpha_i, \beta) = x_i \end{aligned}$$

and

$$\begin{aligned} & \mathbb{P}^{(i)}\left(\left|\prod(\beta')\right|_{\alpha_1}, \cdots, \left|\prod(\beta')\right|_{\alpha_l}\right) = \mathbb{P}^{(i)}(T(\beta')) = \mathbb{P}(\vec{\mathbf{Z}}_1^{(i)} = T(\beta')) \\ = \quad & f(\alpha_i, \beta') = 1 - x_i \end{aligned}$$

and hence $\mathbf{R} = \{\mathbf{p}_{\alpha_i}^{(1)}\}_{i=1}^{l} = \mathbf{S}_R(\vec{x})$ is a transition model from $\mathbf{S}$ to $\mathbf{S}'$. $\square$

Therefore, if $\mathbf{M}$, $\mathbf{M}'$ and $\mathbf{M}(\vec{x})$ are the ξ-matrices and mean offspring matrix of the models $\mathbf{S}$, $\mathbf{S}'$ and $\mathbf{S}_R(\vec{x})$ with spectral radii ρ, ρ' and $\rho(\vec{x})$, respectively, then, by Theorem 3.1.2 in Ban et al (2021), we have the following propositions.

Proposition 24. *Under the assumptions in Proposition 21 and under the sup norm for the matrices and vectors, we have*

(i) *If* $\vec{x} \to \vec{0}$*, then* $\rho(\vec{x}) \to \rho$*.*
(ii) *If* $\vec{x} \to \vec{1}$*, then* $\rho(\vec{x}) \to \rho'$*.*

According to Lemma 13 in Section 3.1.2, we have that, in the induced system, the spread rate of the transition model $\mathbf{S}_R(\vec{x})$ gradually changes from the spread rate of $\mathbf{S}$ to that of $\mathbf{S}'$ as $\vec{x} \to \vec{1}$. Since the m-spread model shares the same spread rate with its induced model, the spread rate of the transition model $\mathcal{S}_R(\vec{x})$ also gradually changes from the spread rate of the m-spread model $\mathcal{S}$ to that of $\mathcal{S}'$ as $\vec{x} \to \vec{1}$. Moreover, we have the following proposition:

Proposition 25. *Under the sup norm for matrices and vectors, the map $\vec{x} \to \rho(\vec{x})$ is continuous.*

5. Numerical results

This section presents several examples that numerically demonstrate or verify the validity of the previous theorems.

5.1. **Topological models.** The experiment below provides an evidence for the validity of the formula stated in Theorem 4. Let $\mathcal{A}$ be the type set consists of two symbols a and b, and let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathcal{P}_1$ be 1-patterns defined as

$$\begin{aligned} \alpha_1 &= (a; a, a, a, b), \\ \alpha_2 &= (a; a, a, b, b, b), \\ \alpha_3 &= (b; a, b, b, b, b) \\ \alpha_4 &= (b; b, b, b, b, b). \end{aligned}$$

Consider a 2-spread model $\mathcal{S} = \{p_1, p_2, p_3, p_4\}$ defined by its associated induced 1-order spread model $\mathbf{S} = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4\}$ over type set $\mathbf{A} = \{p_1, p_2, p_3, p_4\}$ (more specifically, $\mathbf{p}_i$ corresponds to p_i), where

$$\begin{cases} \mathbf{p}_1 = (\alpha_1; \alpha_1, \alpha_2, \alpha_2, \alpha_3), \\ \mathbf{p}_2 = (\alpha_2; \alpha_1, \alpha_2, \alpha_4, \alpha_4, \alpha_4), \\ \mathbf{p}_3 = (\alpha_3; \alpha_1, \alpha_3, \alpha_4, \alpha_4, \alpha_4), \\ \mathbf{p}_4 = (\alpha_4; \alpha_3, \alpha_3, \alpha_4, \alpha_4, \alpha_4), \end{cases} \tag{19}$$

and let $k_n = 2 \cdot \lfloor \log n + 1 \rfloor$ so that s_n is specified according to (2). One can easily compute the ξ-matrix $\mathbf{M}$ associated with model $\mathbf{S}$, which is

$$\mathbf{M} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

with $\mathbf{v} = [0.0905, 0.0463, 0.3075, 0.5558]^T$ its probability right eigenvector. By virtue of Theorem 5, we know that

$$s_p(p_i; \{k_n\}_{n=1}^{\infty}) = s_{\mathbf{p}}(\mathbf{p}_i; \{k_n\}_{n=1}^{\infty}) = \lim_{n \to \infty} s_{\mathbf{p}}(\mathbf{p}_i; [s_n, s_{n+1}]),$$

in which the convergence of the limit is reassured in Figure 8. Then, consistent with Theorem 4, the ratios $s_p(a; [s_n, s_{n+1}])$ and $s_p(b; [s_n, s_{n+1}])$ of symbols a and b converge to the corresponding sums of all associated entries in the right eigenvector $\sum_{\eta \in \theta(a)} v(\eta) = v(p_1) + v(p_2) = 0.1368$ and $\sum_{\eta \in \theta(b)} v(\eta) = v(p_3) + v(p_4) = 0.8632$, as is seen in Figure 9.

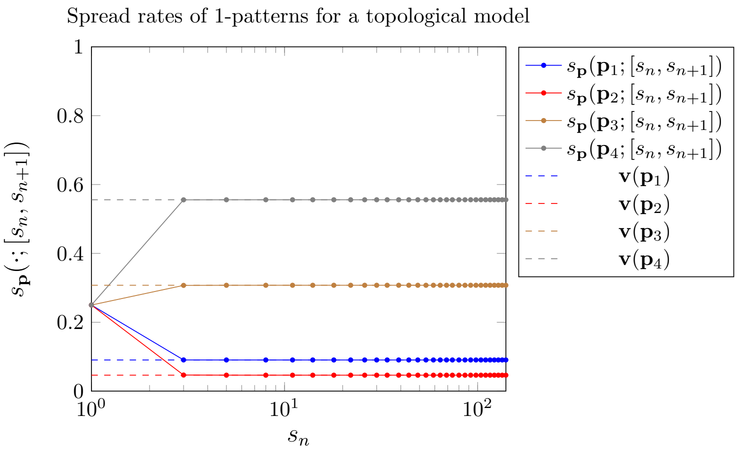


FIGURE 8. Spread rates $s_p(p_i; [s_n, s_{n+1}]) = s_{\mathbf{p}}(\mathbf{p}_i; [s_n, s_{n+1}])$ of the topological 2-spread model defined in (19). The ratios $s_{\mathbf{p}}(\mathbf{p}_i; [s_n, s_{n+1}])$ are observed to approach $\mathbf{v}(\mathbf{p}_i)$ in the figure.

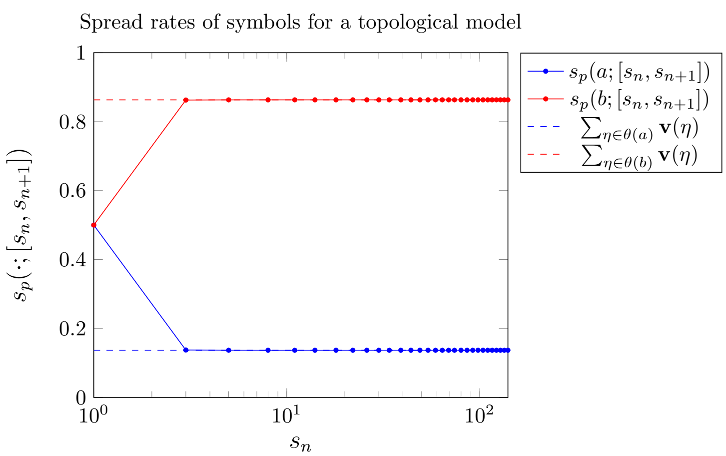


FIGURE 9. Spread rates $s_p(a; [s_n, s_{n+1}])$ and $s_p(b; [s_n, s_{n+1}])$ of the topological 2-spread model defined in (19). The ratios $s_p(a; [s_n, s_{n+1}])$ and $s_p(b; [s_n, s_{n+1}])$ are observed to approach $\mathbf{v}(\mathbf{p}_1) + \mathbf{v}(\mathbf{p}_2)$ and $\mathbf{v}(\mathbf{p}_3) + \mathbf{v}(\mathbf{p}_4)$, respectively, in the figure.

5.2. **Random models.** The experiments in this subsection support Theorem 15 numerically. For the sake of comparison, we consider here a special case of random 2-spread model that is generated by a random 1-spread model. Let $\mathcal{A} = \{a, b\}$ again be the type set and $k_n = 2 \cdot \lfloor \log n + 1 \rfloor$ so that s_n is specified according to

(2). Then, we set $\mathcal{Q}_1 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where

$$\begin{cases} \alpha_1 = (a; a, a, a, b, b), \\ \alpha_2 = (a; a, a, b, b, b), \\ \alpha_3 = (b; a, b, b, b, b), \\ \alpha_4 = (b; b, b, b, b, b, b), \end{cases}$$

upon which we choose the pattern distribution function $f_0 : \mathcal{Q}_1^{(0)} \times \mathcal{Q}_1 \to [0, 1]$ to be

$$f_0(a, \alpha_i) = \begin{cases} 0.6, & \text{if } i = 1; \\ 0.4, & \text{if } i = 2; \\ 0, & \text{otherwise;} \end{cases} \qquad f_0(b, \alpha_i) = \begin{cases} 0.5, & \text{if } i = 3; \\ 0.5, & \text{if } i = 4; \\ 0, & \text{otherwise,} \end{cases}$$

so one derives via direct computation the associated offspring mean matrix

$$\mathbf{M}_0 = \begin{bmatrix} 2.6 & 0.5 \\ 2.4 & 5.0. \end{bmatrix}$$

and its right eigenvector $\mathbf{v}_0 = [0.1504, 0.8496]^T$. We note this random 1-spread model naturally generates a random 2-spread model $\mathcal{R}$ that is previously seen as a byproduct in the induced branching process (Section 3.1.2). More explicitly, we define $\mathcal{Q}_2^{(0)} = \mathcal{Q}_1$ and, following the convention of (1),

$$\mathcal{Q}_2 = \{(\beta; \gamma_1, \cdots, \gamma_{\sigma_\beta^{(1)}}) : \beta, \gamma_i \in \mathcal{Q}_1, \beta_i^{(1)} = \gamma_i^{(0)}\}, \tag{20}$$

for which a natural pattern distribution function $f : \mathcal{Q}_2^{(0)} \times \mathcal{Q}_2 \to [0, 1]$ of the random 2-spread model $\mathcal{R}$ is given by

$$f(\alpha, \boldsymbol{\alpha}) = \begin{cases} f_0(\alpha^{(0)}, \boldsymbol{\alpha}) \cdot \prod_{i=1}^{\sigma_\alpha^{(1)}} f_0((\boldsymbol{\alpha}_i^{(1)})^{(0)}, \boldsymbol{\alpha}_i^{(1)}), & \text{if } \boldsymbol{\alpha}^{(0)} = \alpha; \\ 0, & \text{otherwise;} \end{cases} \tag{21}$$

and the associated offspring mean matrix can be computed to be

$$\mathbf{M} = \begin{bmatrix} 1.8 & 1.2 & 0.6 & 0 \\ 1.2 & 0.8 & 0.4 & 0 \\ 1 & 1.5 & 2. & 3 \\ 1 & 1.5 & 2. & 3 \end{bmatrix}$$

and $\mathbf{v} = [0.0902, 0.0602, 0.4248, 0.4248]^T$ is the associated eigenvector. If we apply Theorem 17 with $r = 2$, then almost surely

$$\lim_{n\to\infty} \frac{\mathcal{O}_{\alpha_i}(\tau_{\alpha_j}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_j})})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_j})\right|} = \sum_{\alpha_i \in \theta_1(\alpha_j)} \mathbf{v}(\alpha_i) = \mathbf{v}(\alpha_j).$$

This agrees with the numerical simulation plotted in Figure 10 for the case $j = 1$, in which each marked point is taken to be the average over 30 realizations. On the other hand, Theorem 15 asserts that almost surely

$$\lim_{n\to\infty} \frac{\mathcal{O}_{a}(\tau_{\alpha_1}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_1})})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_1})\right|} = \sum_{\alpha_i \in \theta_1(a)} \mathbf{v}(\alpha_i) = \mathbf{v}(\alpha_1) + \mathbf{v}(\alpha_2)$$

and

$$\lim_{n\to\infty} \frac{\mathcal{O}_{b}(\tau_{\alpha_1}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_1})})}{\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_1})\right|} = \sum_{\alpha_i \in \theta_1(b)} \mathbf{v}(\alpha_i) = \mathbf{v}(\alpha_3) + \mathbf{v}(\alpha_4).$$

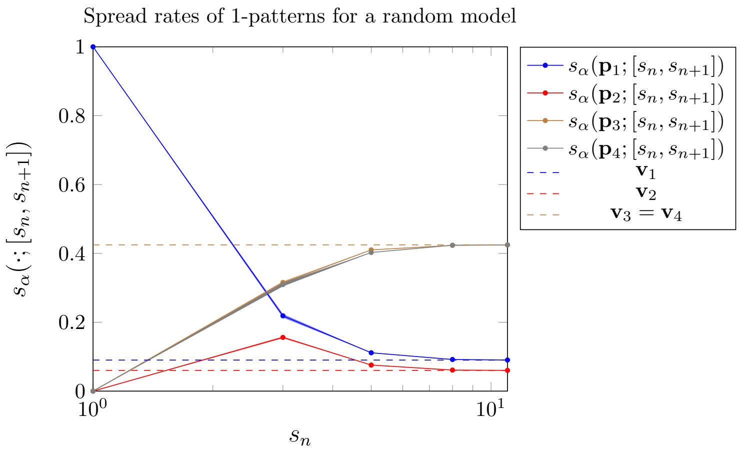


FIGURE 10. Spread rates $\mathcal{O}_{\alpha_i}(\tau_{\alpha_1}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_1})})/\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_1})\right|$ of the topological 2-spread model defined by (20) and (21). The simulation begins with root α_1 and each data point is taken to be the average over 30 realizations. The ratios are observed to approach $\mathbf{v}(\alpha_i)$ in the figure.

This is captured in Figure 11, in which each marked point is again taken to be the average over 30 realizations. Furthermore, this is also consistent with Theorem 17 with $r = 1$ by noting that $\mathbf{v}(\alpha_1) + \mathbf{v}(\alpha_2) = \mathbf{v}_0(a)$ and that $\mathbf{v}(\alpha_3) + \mathbf{v}(\alpha_4) = \mathbf{v}_0(b)$.

5.3. **Relations between topological and random models.** Let $\mathcal{A} = \{a, b\}$, $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$ be as defined in (19), and $\mathbf{p}_1' = (\alpha_1; \alpha_1, \alpha_2, \alpha_3)$. We define topological 2-spread models

$$\mathcal{S} = \{p_1, p_2, p_3, p_4\} \quad \text{and} \quad \mathcal{S} = \{p_1', p_2, p_3, p_4\} \tag{22}$$

as well as the transition model $\mathcal{S}_R(x_1, x_2, x_3, x_4)$. In fact, the transition model is, by definition, a model depending only on x_1. Thus, according to Proposition 24, the corresponding spectral radius $\rho(x_1, x_2, x_3, x_4) = \rho(x_1)$ is continuous with respect to x_1. This is observed in Figure 12.

6. CONCLUSION

When a pandemic persists for a prolonged period, the spread of the infectious disease becomes increasingly complicated. From many research reports, it is clear that once a person is infected, he or she may become another source of infection, even during the incubation and recovery periods. To describe this phenomenon, we propose two mathematical models from the topological and random perspectives by means of substitution dynamical systems and the theory of branching processes. In both proposed models, the type structure of the current generation or at the current time depends not only on the type structures of the previous generation but also on the type structures of the past m generations. Therefore, they are called the topological and random m-spread model, respectively.

In this work, we construct a corresponding induced system and induced branching process for the m-spread model, apply the classical results from substitution

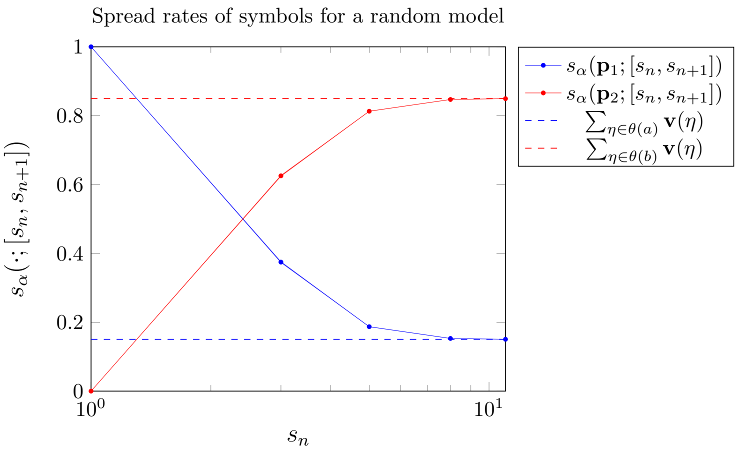


FIGURE 11. Spread rates $\mathcal{O}_a(\tau_{\alpha_1}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_1})})/\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_1})\right|$ and $\mathcal{O}_b(\tau_{\alpha_1}|_{\Delta_{s_n}^{s_{n+1}}(\tau_{\alpha_1})})/\left|\Delta_{s_n}^{s_{n+1}-r}(\tau_{\alpha_1})\right|$ of the topological 2-spread model defined by (20) and (21). The simulation begins with root α_1 and each data point is taken to be the average over 30 realizations. The two aforementioned ratios are observed to approach $\mathbf{v}(a)=\mathbf{v}(\alpha_1)+\mathbf{v}(\alpha_2)$ and $\mathbf{v}(a)=\mathbf{v}(\alpha_3)+\mathbf{v}(\alpha_4)$, respectively, in the figure.

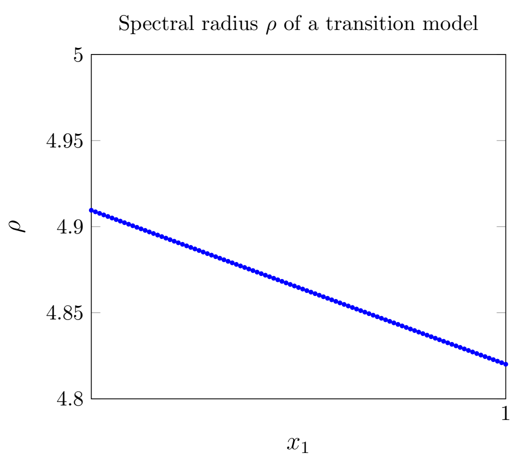


FIGURE 12. Spectral radius $\rho(x_1,x_2,x_3,x_4)$ of the transition model defined by (22). The continuous of the function is observed in the figure.

dynamical systems and the theory of branching processes, and use their matrix representations to established a computable method to predict the long-term spread rate of a type (or a symbol) as well as a pattern within some given range of generations. As an application, we also draw a comparison between two spread models with different initial spread patterns. Moreover, the connection between the topological m-spread model and the random m-spread model is analyzed and some numerical results are provided at the end of this paper.

Acknowledgment

The authors sincerely thank the anonymous reviewers for the valuable comments which helped improve and clarify the manuscript.

Data availability

All data generated or analysed during this study are included in this published article.

References

Al-J螺ararha M, Ou CH (2013) Population dynamics with age-dependent diffusion and death rates. European Journal of Applied Mathematics 24(4):471–500

Alexander ME, Bowman C, Moghadas SM, et al (2004) A vaccination model for transmission dynamics of influenza. SIAM Journal on Applied Dynamical Systems 3(4):503–524

Athreya KB, Ney PE (2004) Branching processes. Courier Corporation

Ban JC, Chang CH, Hong JI, et al (2021) Mathematical analysis of spread models: From the viewpoints of deterministic and random cases. Chaos, Solitons & Fractals 150:111,106

Berger T, Ye ZX (1990) Entropic aspects of random fields on trees. IEEE Transactions on Information Theory 36(5):1006–1018

Billings L, Bollt EM, Schwartz IB (2002) Phase-space transport of stochastic chaos in population dynamics of virus spread. Physical Review Letters 88(23):234,101

Gourley SA, Lou YJ (2014) A mathematical model for the spatial spread and biocontrol of the Asian longhorned beetle. SIAM Journal on Applied Mathematics 74(3):864–884

Halle KS, Wu CW, Itoh M, et al (1993) Spread spectrum communication through modulation of chaos. International Journal of Bifurcation and Chaos 3(02):469–477

Jin Y, Zhao XQ (2009) Spatial dynamics of a periodic population model with dispersal. Nonlinearity 22(5):1167

Lind D, Marcus B (1995) An introduction to symbolic dynamics and coding. Cambridge University Press, Cambridge

Merkin JH, Needham DJ, Sleeman BD (2005) A mathematical model for the spread of morphogens with density dependent chemosensitivity. Nonlinearity 18(6):2745

Ou CH, Wu JH (2006) Spatial spread of rabies revisited: influence of age-dependent diffusion on nonlinear dynamics. SIAM Journal on Applied Mathematics 67(1):138–163

Queffélec M (2010) Substitution dynamical systems-spectral analysis, vol 1294. Springer, lecture Notes in Mathematics

Ruan SG, Wu JH (2009) Modeling spatial spread of communicable diseases involving animal hosts. Spatial ecology pp 293–316

Wang WD, Zhao XQ (2012) Basic reproduction numbers for reaction-diffusion epidemic models. SIAM Journal on Applied Dynamical Systems 11(4):1652–1673

Yang WG, Liu W (2002) Strong law of large numbers and Shannon-McMillan theorem for Markov chain fields on trees. IEEE Transactions on Information Theory 48(1):313–318. https://doi.org/10.1109/18.971762

Yang WG, Ye ZX (2007) The asymptotic equipartition property for nonhomogeneous Markov chains indexed by a homogeneous tree. IEEE Transactions on Information Theory 53(9):3275–3280

Ye ZX, Berger T (1996) Ergodic, regularity and asymptotic equipartition property of random fields on trees. Journal of Combinatorics, Information & System Sciences 21:157–184

Zheng H, Bonasera A (2020) Chaos, percolation and the coronavirus spread: a two-step model. The European Physical Journal Plus 135(10):1–24

(Jung-Chao Ban) Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan, ROC.

Math. Division, National Center for Theoretical Science, National Taiwan University, Taipei 10617, Taiwan. ROC.
Email address: jcban@nccu.edu.tw

(Jyy-I Hong) Department of Mathematical Sciences, National Chengchi University, Taipei 11605, Taiwan, ROC.
Email address: hongjyyi@nccu.edu.tw

(Yu-Liang Wu) Department of Mathematical Sciences, P.O. Box 3000, 90014 University of Oulu, Finland
Email address: yu-liang.wu@oulu.fi