AN ANALOGUE OF TOPOLOGICAL SEQUENCE ENTROPY FOR MARKOV HOM TREE-SHIFTS

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Abstract. In this article, an analogue of topological sequence entropy in a Markov hom tree-shifts, say h^S_{top}, is defined. We explore various aspects of h^S_{top} over tree-shifts, a quantity originally proposed for the characterization of topological dynamical systems and analogously defined in this work. These aspects include the existence of the limit in topological sequence entropy, its relationship with topological entropy, a full characterization of null systems (systems with zero h^S_{top} for any sequences), and the upper bound as well as denseness of all possible values. A short discussion is also given in the article regarding the relationship between this quantity and a variant called induced entropy. These results as a whole provide an overview of the h^S_{top}.

Contents

2020 *Mathematics Subject Classification.* 37B10, 37E25.

Key words and phrases. Markov hom tree-shifts, topological entropy, topological sequence entropy, topological surface entropy.

Ban and Chang are partially supported by the Ministry of Science and Technology, ROC (Contract No MOST 109-2115-M-004-002-MY2 and 109-2115-M-390-003-MY3) and National Center for Theoretical Sciences. Hu is partially supported by the National Natural Science Foundation of China (Grant No.11601355).

1. Introduction

1.1. **Motivations.** Let $\mathcal{A} = \{0, 1, \ldots, m-1\}$ be a finite set and $\mathcal{T}$ be infinite, locally finite, connected graph without loops and with a distinguished point vertex ϵ. A *full shift* $\mathcal{A}^{\mathcal{T}}$ on $\mathcal{T}$ is the set of all functions $x : \mathcal{T} \to \mathcal{A}$. Suppose A is a $m \times m$ binary matrix indexed by $\mathcal{A}$, a *Markov hom*[1] *tree-shift*, say $\mathcal{T}_A$, is the set consisting of all $x \in \mathcal{A}^{\mathcal{T}}$ with $A_{x_{\widetilde{u}}, x_u} = 1$, where $\widetilde{u}$ is the parent of u. The concept of $\mathcal{T}_A$ was first introduced by [1], and refers to a sort of shift space defined on free semigroups (cf. [20, 7]). The topological behavior and entropy theory has been extensively studied during the last decade, see (cf. [2, 3]) for the topological behavior, ([1]) for the classification theory and (cf. [4, 24, 25]) for the entropy theory. It is worth noting that since the amenability is no longer true for $\mathcal{T}$, that is $|\Delta_n \backslash \Delta_{n-1}| / |\Delta_n|$ does not tend to 0 as $n \to \infty$ (cf. [7]), where Δ_n denotes the set of all vertices of $\mathcal{T}$ whose distance from 0 is at most n, $\mathcal{T}_A$ has an abundant and interesting phenomena which is different from the shift spaces defined on $\mathbb{N}$ or on amenable groups.

The notion of *sequence entropy* was first introduced by Kushnirenko [19], and is an useful invariant which has close relationships with the topological behaviors of a measurable dynamical systems ([27, 21, 11, 12, 23, 14, 16, 13]). Precisely, suppose $f \in C(X)$ and X is a compact metric space. Let $S = \{s_n\}_{n=1}^{\infty}$ be a sequence of natural numbers and α is an open cover of X, denote by $f^{-1}\alpha = \{f^{-1}A : A \in \alpha\}$, and define

$$h_{top}^S(f, \alpha) = \limsup_{n\to\infty} \frac{1}{n} \log N(f^{-s_1}\alpha \vee \ldots \vee f^{-s_n}\alpha), \tag{1}$$

where $N(\alpha)$ is the minimum cardinality of the cover α and $\alpha \vee \beta = \{A \cap B : A \in \alpha, B \in \beta\}$. The *topological sequence entropy* of f with respect to S is defined by

$$h_{top}^S(f) = \sup_{\alpha \in \mathcal{OC}(X)} h_{top}^S(f, \alpha),$$

where $\mathcal{OC}(X)$ is the collection of all open covers of X. If $(X, \mathcal{B}, \mu, f)$ is a measure-theoretical dynamical system, the *measure-theoretical sequence entropy* $h_\mu^S(f)$ can be defined in the same way by replacing $N(\alpha)$ with $H_\mu(\alpha) = -\sum_{A\in\alpha} \mu(A) \log \mu(A)$. The general theme is that several mixing concepts can be investigated via sequence entropy both in measure-theoretical and in topological dynamical systems [14]. We also emphasize that such concept is equivalent to the study of the nonautonomous dynamical systems (NDS) (see (2)) mentioned by Kolyada and Snoha [17]

$$(X, f_1, f_2, f_3, \cdots) := (X; f^{s_1}, f^{s_2-s_1}, f^{s_3-s_2}, \cdots). \tag{2}$$

1.2. **An analogue of topological sequence entropy for Markov hom tree-shifts.** The aim of this article is to extend the notion of topological sequence entropy of a symbolic dynamical system in multidimensional lattices $\mathcal{T}$. First, we provide the definition for the topological entropy of $\mathcal{T}_A$ below. To clarify, we restrict ourselves to the conventional 2-tree in this study.

Let $\Sigma = \{0, 1\}$ and let Σ^* be the set of words over Σ. More specifically, $\Sigma^* = \cup_{n\geq 0} \Sigma^n$, where $\Sigma^n = \{w_1 w_2 \cdots w_n : w_i \in \Sigma \text{ for } 1 \leq i \leq n\}$ is the set of words of length $n \in \mathbb{N}$ and $\Sigma^0 = \{\epsilon\}$ consists of the empty word ϵ. A subset of words $L \subset \Sigma^*$ is called *prefix-closed* if each prefix of L belongs to L. A function u defined on a finite prefix-closed subset L with codomain $\mathcal{A}$ is called a *pattern* (or *block*), and L

[1] The word 'hom' indicates that the rule, i.e., the matrix A, on every generator of the tree $\mathcal{T}$ is the same. We refer the reader to [8] for the definition on $\mathbb{Z}^d$ and complete bibliography there.

is called the *support* of the pattern. For $n \in \mathbb{N}$, let $\Delta_n = \cup_{k=0}^{n} \Sigma^k$ denote the set of words of length at most n. We say that a pattern u is a *block of height* n (or an n-*block*) if the support of u is Δ_n, denoted by height$(u) = n$. Let $\mathcal{T}_A$ be a Markov hom tree-shift, we denote by $B_n(\mathcal{T}_A)$ the set of n-blocks in $\mathcal{T}_A$. The *topological entropy* of $\mathcal{T}_A$ is defined as

$$h_{top}(\mathcal{T}_A) = \lim_{n\to\infty} \frac{\log |B_n(\mathcal{T}_A)|}{|\Delta_n|}. \tag{3}$$

Note that the limit (3) defining $h_{top}(\mathcal{T}_A)$ exists [24] and is actually the infimum of $\frac{\log|B_n(\mathcal{T}_A)|}{|\Delta_n|}$ [25]. For $\mathcal{T}$ being a golden-mean tree[2], the limit of (3) also exists, and more general results for the existence of the limit (3) can be found in [5].

Let $P \subset \Sigma^*$ be a subset of words, P is called a *prefix set* if no word in P is a prefix of of another one. The maximum length (resp. minimum length) of P, denoted by $l_{\max}(P)$ (resp. $l_{\min}(P)$), is the length of the longest (resp. shortest) word in P. A finite prefix set P is called a *complete prefix code* (CPC) if any $x \in \Sigma^*$, such that $|x| \geq |P|$, has a prefix in P. Suppose $\mathcal{P} = \cup_{n\geq 0} P_n$ is a collection of CPCs, and we call $\mathcal{P}$ *regular* if $l_{\max}(P_n) \leq l_{\min}(P_{n+1})$ for $n \in \mathbb{N}$. In what follows, we assume $\mathcal{P} = \cup_{n\geq 0} P_n$ is regular. Let F be a finite subset of $\mathcal{T}$, we denote by $\mathcal{P} + F := \cup_{n\geq 0}\{P_n + F\}$. For $m \in \mathbb{N}$, we set

$$\mathcal{T}_A^{\mathcal{P};m} = \{y \in \mathcal{A}^{\mathcal{P}+\Delta_m} : y = x|_{P_n+\Delta_m}, \forall n \in \mathbb{N} \cup \{0\} \text{ and for some } x \in \mathcal{T}_A\}.$$

Set $\mathcal{P}_n = \cup_{i=0}^{n-1} P_i$ and $\mathcal{P}_n + \Delta_m = \cup_{i=0}^{n-1} (P_i + \Delta_m)$. The number of the n-blocks on $\mathcal{P} + \Delta_m$ is denoted by $B_n(\mathcal{T}_A^{\mathcal{P};m})$, that is, $B_n(\mathcal{T}_A^{\mathcal{P};m}) = \{x|_{\mathcal{P}_n+\Delta_m} : x \in \mathcal{T}_A\}$. We define an analogue of the topological sequence entropy for a Markov hom tree-shift $\mathcal{T}_A$ on $\mathcal{T}$ below.

$$\begin{aligned} h_{top}(\mathcal{T}_A^{\mathcal{P};m}) &= \limsup_{n\to\infty} \frac{\log \left|B_n(\mathcal{T}_A^{\mathcal{P};m})\right|}{|\mathcal{P}|} \\ &= \limsup_{n\to\infty} \frac{\log \left|B_n(\mathcal{T}_A^{\mathcal{P};m})\right|}{\Sigma_{i=0}^{n-1}|P_i|}, \end{aligned} \tag{4}$$

and

$$h_{top}(\mathcal{T}_A^{\mathcal{P}}) = \lim_{m\to\infty} h_{top}(\mathcal{T}_A^{\mathcal{P};m}). \tag{5}$$

The intrinsic meaning behind $h_{top}(\mathcal{T}_A^{\mathcal{P};m})$ is that if we project $\mathcal{T}_A^{\mathcal{P};m}$ along some infinite path $\tau = (\epsilon, \tau_{n_0}, \tau_{n_0}\tau_{n_1}, \cdots)$ from $\mathcal{T}$, where $\tau_{n_i} \in \{\tau_0, \tau_1\}$ stands for the generators of $\mathcal{T}$. The value in (4) becomes

$$h_{top}(\mathcal{T}_A^{\mathcal{P};m}|_\tau) = \limsup_{n\to\infty} \frac{\log \left|\{x|_{\cup_{i=0}^{n-1}\cup_{j=0}^{m-1}(s_i+j)} : x \in X_A\}\right|}{n}, \tag{6}$$

where X_A is the subshift of finite type (SFT) with the adjacency matrix A for some $\{s_i\}_{i=0}^{\infty} \subseteq \mathbb{N}$ that is induced from the projection of $\mathcal{P}$ along the path τ. It is worth pointing out that form (6) shares the same concept with the definition of $h_{top}^S(f, \alpha)$ (cf. (1)). Hence, $h_{top}(\mathcal{T}_A^{\mathcal{P};m})$ may be considered an analogue of the topological sequence entropy defined on the multidimensional lattice $\mathcal{T}$. The main

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[2]That is, the relations between the generators of the group $G = \langle S_2 | R\rangle$ are represented by a binary matrix $K = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and $S_2 = \{a, b\}$ (cf. [5]).

purpose of this article is to establish some fundamental properties of $h_{top}(\mathcal{T}_A^{\mathcal{P}})$, and to understand how the lattice $\mathcal{T}$ affects the topological sequence entropy.

As the study of $h_{top}(\mathcal{T}_A^{\mathcal{P}})$ for an arbitrary sequence of CPCs $\mathcal{P}$ is extremely difficult. For clarity, we first restrict ourselves in this article to cases where $\mathcal{P} = \cup_{n\geq 0} P_n$ is "uniform" and $m = 0$ (i.e., the 0-cylinder $\mathcal{A}$). More precisely, given $S = \{s_n\}_{n=0}^{\infty} \subseteq \mathbb{N}$ with $s_0 = 0$ and set[3] $\Sigma_S^* = \cup_{n=0}^{\infty} \Sigma^{s_n}$. Define

$$\mathcal{T}_A^S = \{y \in \mathcal{A}^{\Sigma_S^*} : y = x|_{\Sigma_S^*} \text{ for some } x \in \mathcal{T}_A\}.$$

Similarly, we have $\Delta_n^S = \cup_{i=0}^{n-1} \Sigma^{s_i}$ and $B_n(\mathcal{T}_A^S) = \left\{x|_{\Delta_n^S} : x \in \mathcal{T}_A\right\}$. We define $h_{top}(\mathcal{T}_A^S)$ as

$$h_{top}(\mathcal{T}_A^S) = \limsup_{n\to\infty} \frac{\log \left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|}, \tag{7}$$

and denote it by $h_{top}^S(\mathcal{T}_A) := h_{top}(\mathcal{T}_A^S)$.

1.3. **Main results.** The principal results are presented below.

1. (Fundamental properties of $h_{top}^S(\mathcal{T}_A)$) In Section 2, the fundamental properties of the h_{top}^S are demonstrated. The existence of $h_{top}^S(\mathcal{T}_A)$ is, of course, the first stage of this study. Theorem 2.1 provides the necessary conditions for the existence of $h_{top}^S(\mathcal{T}_A)$ that is associated with the primitive property of A. We also construct a specific $S \subseteq \mathbb{N}$ where $h_{top}^S(\mathcal{T}_A)$ does not exist. In other words, $h_{top}^S(\mathcal{T}_A)$ does not always exist in general.

Recall the works of Newton [22] and Goodman [12]. Let $S = \{s_i\}_{i=0}^{\infty} \subseteq \mathbb{N}$, and $U_S(n,k) = \{s_i + j : 0 \leq i \leq n, 0 \leq j \leq k-1\}$. Define $K(S) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{|U_S(n,k)|}{n}$, where $K(S)$ is well-defined since $\limsup_{n\to\infty} n^{-1} |U_S(n,k)|$ is an increasing non-negative function of k. Let (X, f) be a flow and suppose X has finite covering dimension (see [12]). If $0 < h_{top}(f) < \infty$, Theorem 4.4 [12] indicates that

$$h_{top}^S(f) = K(S) h_{top}(f). \tag{8}$$

Theorem 2.2 gives an analogous result as (8) for $\mathcal{T}_A^S$. However, the equality does not hold in general, and we provide the lower and upper bound for $h_{top}^S(\mathcal{T}_A)$ instead.

Let (X, f) be a topological dynamical system and $\mathcal{U}$ is an open cover of X. Huang et al. [14] proved that if $h_{top}(f, \mathcal{U}) > 0$, then for each infinite sequence $S \subseteq \mathbb{N}$ one has $h_{top}^S(f, \mathcal{U}) > 0$. Under some conditions on S, Theorem 2.3 presents a similar result by showing that

$$h_{top}^S(\mathcal{T}_A) \geq h_{top}(\mathcal{T}_A).$$

Goodman [12] introduced the notion of topological sequence entropy and studied some properties of null systems which are defined as having zero topological sequence entropy for any infinite sequence. It is a natural question whether we have characterization of the null systems for $\mathcal{T}_A^S$. Theorem 2.4 provides such a characterization. We point out that the characterization is for general $\mathcal{T}_X^S$, the tree-shifts on $\mathcal{T}$. Moreover, if $\mathcal{T}_X^S = \mathcal{T}_A^S$, the condition is finitely checkable.

2. (The supremum of sequence entropies over all subsequences of $\mathbb{N}$) Section 3 discusses the supremum of sequence entropies over all subsequences of $\mathbb{N}$, and below is a description of the motivation. Let X be a compact metric space and

[3] That is, $\mathcal{P} = \Sigma_S^*$ and $P_n = \Sigma^{s_n}$ for all $n \in \mathbb{N} \cup \{0\}$.

$f : X \to X$ be continuous. Let $h^{\infty}_{top}(f)$ be the supremum of topological sequence entropies of f over all subsequences of $\mathbb{N}$ and $\mathcal{H}^{\infty}(X)$ be the possible values of $h^{\infty}_{top}(f)$ for all continous maps f on X. The value $h^{\infty}_{top}(f)$ is also known as *maximal pattern entropy* which is introduced in [15]. It is known that

$$\mathcal{H}^{\infty}(X) \subseteq \{\infty, 0, \log 2, \log 3, \ldots\}, \tag{9}$$

and if X is a finite tree or the unit circle S^1, then $\mathcal{H}^{\infty}(X) = \{\infty, 0, \log 2\}$ (cf. [28]). Some related results can also be found in [18]. Similarly, we define

$$h^{\infty}_{top}(\mathcal{T}_A) = \sup_{S \subseteq \mathbb{N}} h^{S}_{top}(\mathcal{T}_A),$$

where $\mathcal{T}_A$ is a tree-shift. Theorem 3.5 demonstrates that $h^{\infty}_{top}(\mathcal{T}_A) \in (\log 4, \log 5)$, that is, (9) is no longer true. We emphasize that Theorem 3.5 is interesting since this phenemena cannot happen in 1-d SFTs[4]. Although (9) is not true for $h^{\infty}_{top}(\mathcal{T}_A)$, Corollary 3.3 shows that $h^{\infty}_{top}(\mathcal{T}_A) = h^{L}_{top}(\mathcal{T}_A)$ and both of them dominate $h_{top}(\mathcal{T}_A)$, where $h^{L}_{top}(\mathcal{T}_A)$ is called the *topological surface entropy* since it has the same intrinsic meaning as the surface entropy[5] introduced by Berger and Ye [6]. Furthermore, we also prove that the set $\{h^{L}(\mathcal{T}_A) : \mathcal{T}_A$ is defined on d-tree and A is irreducible$\}$ is dense in $[d \log 2, \infty)$ (Theorem 3.6). This indicates that the set of possible values of $h^{\infty}_{top}(\mathcal{T}_A)$ is not the discrete set of the logarithm of integers or ∞. On the contrary, they form a dense set of some interval in $\mathbb{R}$. This result is novel and unlike previous results of $h^{\infty}_{top}(f)$, where f is a 1-dimensional map.

3. (The induced entropy) In Section 4, we introduce the concept of the *induce entropy* as follows. For $S = \{s_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$, which is an increasing sequence, the associated sequence $K = \{k_n\}_{n=1}^{\infty}$ is set as $k_n = s_{n+1} - s_n$ for all $n \in \mathbb{N}$, that is, the length of the interval $[s_n, s_{n+1}]$. Note that K is not necessary increasing. The shift $\mathcal{T}^{ind}_A$ induced from $\mathcal{T}_A$ is defined as

$$\mathcal{T}^{S;ind}_A = \{x \in \mathcal{A}^{\Sigma^*} : A^{k_n}_{x_\omega, x_{\omega i}} > 0 \text{ for all } n \geq 0,\ \omega \in \Delta_n,\ i \in \{0,1\}\}. \tag{10}$$

The difference between $\mathcal{T}^{S;ind}_A$ and $\mathcal{T}^{S}_A$ is that $\mathcal{T}^{S}_A$ is a 'projection' of $\mathcal{T}_A$ along Σ^*_S and $\mathcal{T}^{S;ind}_A$ is the set of x in which the symbols of x_ω and $x_{\omega i}$ obey the rule of A^{k_n}, for all $n \geq 0$ $\omega \in \Delta_n$ and $i \in \{0,1\}$. The *induce entropy*, say $h^{S;(i)}_{top}(\mathcal{T}_A) := h_{top}(\mathcal{T}^{S;ind}_A)$ is similarly defined. It is obvious that if $S = \{n\}_{n=0}^{\infty}$ we have $h^{S}_{top}(\mathcal{T}_A) = h^{S;(i)}_{top}(\mathcal{T}_A) = h_{top}(\mathcal{T}_A)$. The reason we define $h^{S;(i)}_{top}(\mathcal{T}_A)$ is because there are more than one way to extend the concept of the h^{S}_{top} to multidimensional lattice $\mathcal{T}$. It is nature to understand the relations between $h^{S}_{top}(\mathcal{T}_A)$ and $h^{S;(i)}_{top}(\mathcal{T}_A)$. Theorem 4.2 indicates that $h^{S;(i)}_{top}(\mathcal{T}_A) \geq h^{S}_{top}(\mathcal{T}_A) \geq h_{top}(\mathcal{T}_A)$ under some assumption, and the above two inequalities are strict (Example 4.3). Finally, Theorem 4.1 introduces a similar characterization as Theorem 2.4 of the null systems for $\mathcal{T}^{ind}_A$.

2. Fundamental properties of $h^{S}_{top}(\mathcal{T}_A)$

2.1. **Existence of $h^{S}_{top}(\mathcal{T}_A)$.** This subsection is devoted to the existence of the limit in the definition of the h^{S}_{top}. From now on, the standing assumption of the

[4]We leave it to the reader to verify that if X_A is an 1-d SFT, then $h^{\infty}_{top}(\sigma_A) \in \{0, \log 2\}$, where σ_A is the standard shift map on X_A.

[5]That is, $h^{L}_{top}(\mathcal{T}_A)$ is the growth rate of the possible number of patterns on the nth level of $\mathcal{T}$ as $n \to \infty$.

discussion is that $S = \{s_n\}_{n=0}^{\infty}$ is an increasing sequence of nonnegative integers and thus the sequence of differences $K = \{k_n\}_{n=0}^{\infty}$, defined as $k_n = s_{n+1} - s_n$, is a nonnegative sequence. Under this assumption, we present the following theorem regarding the existence of $h_{top}^{S}(\mathcal{T}_A)$.

Theorem 2.1. *Suppose A is primitive, i.e., a matrix admitting a positive integer $n \in \mathbb{N}$ such that $A^n > 0$. Then, the following assertions holds.*

1. *If $\lim_{n\to\infty} k_n$ exists, then $h_{top}^{S}(\mathcal{T}_A)$ exists as limit.*
2. *There is a sequence $K = \{k_n\}_{n=0}^{\infty}$ such that $\limsup_{n\to\infty} k_n \geq n_A > \liminf_{n\to\infty} k_n$ and $h_{top}^{S}(\mathcal{T}_A)$ does not exist.*

Proof. We note that the (1) of the theorem is similar to that for the topological entropy, and a proof is given right below. As for (2), it involves the properties of the topological surface entropy, and the proof is postponed to the end of Section 3.2.

We first prove the proposition when S is an arithmetic progression. By denoting $S_r = \{rn\}_{n=0}^{\infty}$, we note that $\mathcal{T}_A^{S_r}$ is itself a tree-shift, for which $h_{top}^{S_r}(\mathcal{T}_A)$ is, by definition, the topological entropy of $\mathcal{T}_A^{S_r}$ and thus the existence of the limit follows from an identical argument as [24, Theorem 2.1]. For general $S = \{s_n\}_{n=0}^{\infty}$ satisfying the assumption of the foregoing proposition, let N be an integer such that $k_n = r$ for all $n \geq N$. For $n \geq N$, we note that each $u \in B_n(\mathcal{T}_A^S)$ is composed of an element in $B_N(\mathcal{T}_A^S)$ rooted at the ϵ and 2^{s_N} elements of $B_n(\mathcal{T}_A^{\{ri\}_{i=0}^{\infty}})$ rooted at elements of Σ^{s_N}. This implies

$$\left|B_n(\mathcal{T}_A^S)\right| \leq \left|B_N(\mathcal{T}_A^S)\right| \left|B_{n-N}(\mathcal{T}_A^{S_r})\right|^{2^{s_N}},$$

and thus

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\log\left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|} \\
&\leq \limsup_{n\to\infty} \frac{\log\left|B_N(\mathcal{T}_A^S)\right|}{|\Delta_n^S|} + \frac{2^{s_N}\cdot\log\left|B_{n-N}(\mathcal{T}_A^{S_r})\right|}{|\Delta_n^S|} \qquad (11)\\
&= \limsup_{n\to\infty} \frac{2^{s_N}\left|\Delta_{n-N}^{S_r}\right|}{|\Delta_n^S|}\cdot\frac{\log\left|B_{n-N}(\mathcal{T}_A^{S_r})\right|}{\left|\Delta_{n-N}^{S_r}\right|} = h_{top}^{S_r}(\mathcal{T}_A)
\end{aligned}$$

On the other hand, by setting $M = \inf_{ri\geq s_N} i$, it is not hard to see that every $u \in B_{n-N}(\mathcal{T}_A^{\{s_{i+N}\}_{i=0}^{\infty}})$ is composed of 2^{rM-s_N} elements in $B_{n-N}(\mathcal{T}_A^{rM+S_r})$. Thus,

$$\left|B_n(\mathcal{T}_A^S)\right| \geq \left|B_{n-N}(\mathcal{T}_A^{\{s_{i+N}\}_{i=0}^{\infty}})\right| \geq \left|B_{n-N}(\mathcal{T}_A^{rM+S_r})\right|^{\frac{1}{2^{rM-s_N}}}.$$

Now that $\left|B_{n-N}(\mathcal{T}_A^{rM+S_r})\right| \left|B_{M-1}(\mathcal{T}_A^{S_r})\right| \geq \left|B_{n-N+M}(\mathcal{T}_A^{S_r})\right|$, the inequality above yields

$$\liminf_{n\to\infty} \frac{\log\left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|} = \liminf_{n\to\infty} \frac{\log\left|B_{n-N}(\mathcal{T}_A^{\{s_{i+N}\}_{i=0}^\infty})\right|}{\left|\Delta_{n-N}^{\{s_{i+N}\}_{i=0}^\infty}\right|}$$

$$\geq \liminf_{n\to\infty} -\frac{\log\left|B_{M-1}(\mathcal{T}_A^{S_r})\right|}{2^{rM-s_N}\left|\Delta_{n-N}^{\{s_{i+N}\}_{i=0}^\infty}\right|} + \frac{\log\left|B_{n-N+M}(\mathcal{T}_A^{S_r})\right|}{2^{rM-s_N}\left|\Delta_{n-N}^{\{s_{i+N}\}_{i=0}^\infty}\right|} \tag{12}$$

$$= \liminf_{n\to\infty} \frac{\log\left|B_{n-N+M}(\mathcal{T}_A^{S_r})\right|}{\left|\Delta_{n-N+M}^{S_r}\right|} = h_{top}^{S_r}(\mathcal{T}_A)$$

Combining inequalities (11)(12), one obtains

$$\lim_{n\to\infty} \frac{\log\left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|} = h_{top}^S(\mathcal{T}_A)$$

and the proof is completed. □

2.2. **Relations between $h_{top}^S(\mathcal{T}_A)$ and $h_{top}(\mathcal{T}_A)$.** Let $0 \leq n,\ k \in \mathbb{N}$, and $m(n) = \max\{i : s_i \leq n\}$. Define

$$\mathbf{U}_S(n,k) \quad = \quad \{\omega\nu : \omega \in \Sigma^{s_i}, 0 \leq i \leq n, |\nu| \leq k\}.$$

Decompose $\mathbf{U}_S(n,k)$ into sets of vertices in consecutive levels as $\mathbf{U}_S(n,k) = \cup_{i=0}^{b(n)} \Delta_{l_i}^{c_i}$, where

$$\Delta_l^c = \{\omega\nu : |\omega| = c,\ \ |\nu| \leq l\},$$

and $b(n)$ is an integer such that $c_{b(n)} + l_{b(n)} \leq n$. Let $\delta_l^c = |\Delta_l^c|$ and $\sigma^s = |\Sigma^s|$, define

$$\mathbf{K}(S) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)} \delta_{l_i}^{c_i}}{\sum_{i=0}^{m(n)} \sigma^{s_i}}. \tag{13}$$

Since $\mathbf{U}_S(n,k)$ is an increasing positive function of k,

$$\mathbf{K}(S,k) := \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)} \delta_{l_i}^{c_i}}{\sum_{i=0}^{m(n)} \sigma^{s_i}}$$

is an increasing non-negative function of k. Thus, $\mathbf{K}(S)$ is well-defined and may be ∞. Define

$$\mathbf{C}(S) = \lim_{n\to\infty} \frac{2^{n+1} - \sum_{i=0}^{m(n)} \sigma^{s_i}}{\sum_{i=0}^{m(n)} \sigma^{s_i}} = \lim_{n\to\infty} \frac{\left|\Delta_n \setminus \cup_{i=0}^{m(n)} \Sigma^{s_i}\right|}{\left|\cup_{i=0}^{m(n)} \Sigma^{s_i}\right|}, \tag{14}$$

whenever the limit exists.

Theorem 2.2. *Given $S = \{s_n\}_{n=0}^\infty$ and $s_n \to \infty$ as $n \to \infty$. Let $\mathbf{K}(S)$ and $\mathbf{C}(S)$ be defined as above, then we have*

$$h_{top}^S(\mathcal{T}_A) \leq \mathbf{K}(S) h(\mathcal{T}_A).$$

Furthermore, if $\mathbf{K}(S)$, $\mathbf{C}(S) < \infty$ and A is primitive, then we have

$$h_{top}^S(\mathcal{T}_A) \geq \mathbf{K}(S) h(\mathcal{T}_A) - \mathbf{C}(S) \log 2.$$

Proof. **1**. We first prove that $h_{top}^{S}(\mathcal{T}_A) \leq \mathbf{K}(S)h(\mathcal{T}_A)$. Since

$$h := h(\mathcal{T}_A) = \lim_{n\to\infty} \frac{\log |B_n(\mathcal{T}_A)|}{2^{n+1}}$$

exists, for any $\varepsilon > 0$, there exists an integer $N > 0$ such that $\left|\frac{\log|B_l(\mathcal{T}_A)|}{2^{l+1}} - h\right| \leq \varepsilon$ for $l \geq N$, i.e., for $l \geq N$, we have

$$|B_l(\mathcal{T}_A)| \leq e^{2^{l+1}(h+\epsilon)}. \tag{15}$$

Choose n and k large enough such that $\mathbf{U}_S(n,k) = \sum_{i=0}^{b(n)} \Delta_{l_i}^{c_i}$ and $l_i \geq N$ for $0 \leq i \leq b(n)$. For a finite set F of $\mathcal{T}$, denote by

$$\mathcal{T}_A|_F = \{w \in \{0,1\}^F : w = t|_F \text{ where } t \in \mathcal{T}_A\}$$

the set of the projection of $\mathcal{T}_A$ with respect to $F \subseteq \mathcal{T}$. It is clear that if $E \subseteq F \subseteq \mathcal{T}$, we have $\mathcal{T}_A|_E \subseteq \mathcal{T}_A|_F$ and thus $|\mathcal{T}_A|_E| \leq |\mathcal{T}_A|_F|$. Note that $\{c_i\}_{i=1}^{b(n)} \subseteq \{s_i\}_{i=1}^{m(n)}$ for all $n \in \mathbb{N}$, it follows that for all $0 \leq i \leq m(n)$, $\Delta_k^{s_i}$ is a subset of $\Delta_{l_j}^{c_j}$ for some $0 \leq j \leq b(n)$. Therefore,

$$\left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Sigma^{s_i}}\right| \leq \left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Delta_k^{s_i}}\right| \leq \left|\mathcal{T}_A|_{\cup_{i=0}^{b(n)} \Delta_{l_i}^{c_i}}\right|.$$

Since $\Delta_{l_i}^{c_i}$ and $\Delta_{l_j}^{c_j}$ are nonoverlapping for $0 \leq i \neq j \leq b(n)$, we have

$$\left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Sigma^{s_i}}\right| \leq \left|\mathcal{T}_A|_{\cup_{i=0}^{b(n)} \Delta_{l_i}^{c_i}}\right| \leq \prod_{i=0}^{b(n)} |B_{l_i}(\mathcal{T}_A)|^{2^{c_i}}. \tag{16}$$

Combining (15) and (16) we have

$$\left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Sigma^{s_i}}\right| \leq \prod_{i=0}^{b(n)} e^{2^{(c_i+l_i+1)}(h+\varepsilon)}.$$

Thus,

$$\frac{\log \left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Sigma^{s_i}}\right|}{\sum_{i=0}^{m(n)} \sigma^{s_i}} \leq \frac{\sum_{i=0}^{b(n)} 2^{c_i+l_i+1}}{\sum_{i=0}^{m(n)} \sigma^{s_i}} (h+\varepsilon).$$

Since $\mathbf{K}(S)$ is the limit of increasing sequence $\mathbf{K}(S,k)$, we have

$$\begin{aligned}
h_{top}^{S}(\mathcal{T}_A) &= \limsup_{n\to\infty} \frac{\log \left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)} \Sigma^{s_i}}\right|}{\sum_{i=0}^{m(n)} \sigma^{s_i}} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)} 2^{c_i+l_i+1}}{\sum_{i=0}^{m(n)} \sigma^{s_i}} (h+\varepsilon) \\
&= \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)} \left|\Delta_{l_i}^{c_i}\right|}{\sum_{i=0}^{m(n)} \sigma^{s_i}} (h+\varepsilon) \\
&= \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)} \delta_{l_i}^{c_i}}{\sum_{i=0}^{m(n)} \sigma^{s_i}} (h+\varepsilon) \\
&= \mathbf{K}(S,k)(h+\varepsilon) \\
&\leq \mathbf{K}(S)(h+\varepsilon).
\end{aligned}$$

Since ε is arbitrary, we have $h_{top}^{S}(\mathcal{T}_A) \leq \mathbf{K}(S)h_{top}(\mathcal{T}_A)$.

2. Suppose $\mathbf{K}(S)$ and $\mathbf{C}(S) < \infty$, we claim that

$$h_S(\mathcal{T}_A) \geq \mathbf{K}(S) h(\mathcal{T}_A) - \mathbf{C}(S)\log 2.$$

Recall that ϵ is the root of the tree $\mathcal{T}$. For $a \in \mathcal{A}$ we denote by

$$B_n^a(\mathcal{T}_A) = \{w \in B_n(\mathcal{T}_A) : w(\epsilon) = a\}.$$

For $0 \leq l \in \mathbb{N}$ we set $\underline{a}(l)$ the symbol of $\mathcal{A}$ such that $\left|B_l^{\underline{a}(l)}(\mathcal{T}_A)\right| \leq |B_l^a(\mathcal{T}_A)|$ for all $a \in \mathcal{A}$. Since A is primitive, it follows from [24] that

$$\lim_{n\to\infty} \frac{\log |B_n^a(\mathcal{T}_A)|}{2^{n+1}} = h(\mathcal{T}_A),$$

for all $a \in \mathcal{A}$. Let $N \in \mathbb{N}$ be the same as above, we have $|B_l^a(\mathcal{T}_A)| \geq e^{2^{(l+1)}(h-\varepsilon)}$, for all $l \geq N$. Let n and k large enough such that $U_S(n,k) = \sum_{i=0}^{b(n)} \Delta_{l_i}^{c_i}$ and $l_i \geq N$ for $0 \leq i \leq b(n)$. Since

$$\left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)}\Sigma^{s_i}}\right| 2^{\left(2^{n+1}-\Sigma_{i=0}^{m(n)}\sigma^{s_i}\right)} \geq \left|\mathcal{T}_A|_{\cup_{i=0}^{b(n)}\Delta_{l_i}^{c_i}}\right| \geq \prod_{i=0}^{b(n)} \left|B_{l_i}^{\underline{a}(l_i)}(\mathcal{T}_A)\right|^{2^{c_i}}.$$

Thus,

$$\begin{aligned}
\frac{\log \left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)}\Sigma^{s_i}}\right| 2^{\left(2^{n+1}-\Sigma_{i=0}^{m(n)}\sigma^{s_i}\right)}}{\sum_{i=0}^{m(n)}\sigma^{s_i}} &\geq \sum_{i=0}^{b(n)} 2^{c_i} \frac{\log\left|B_{l_i}^{\underline{a}(l_i)}(\mathcal{T}_A)\right|}{\sum_{i=0}^{m(n)}\sigma^{s_i}} \\
&\geq \frac{\sum_{i=0}^{b(n)} 2^{c_i+l_i+1}}{\sum_{i=0}^{m(n)}\sigma^{s_i}}(h-\varepsilon) \\
&= \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)}\left|\Delta_{l_i}^{c_i}\right|}{\sum_{i=0}^{m(n)}\sigma^{s_i}}(h-\varepsilon) \\
&= \limsup_{n\to\infty} \frac{\sum_{i=0}^{b(n)}\delta_{l_i}^{c_i}}{\sum_{i=0}^{m(n)}\sigma^{s_i}}(h-\varepsilon).
\end{aligned}$$

Since $\mathbf{K}(S)$, $\mathbf{C}(S) < \infty$, we have

$$\begin{aligned}
h_{top}(\mathcal{T}_A^S) &= \limsup_{n\to\infty} \frac{\log\left|\mathcal{T}_A|_{\cup_{i=0}^{m(n)}\Sigma^{s_i}}\right|}{\sum_{i=0}^{m(n)}\sigma^{s_i}} \\
&\geq \mathbf{K}(S)(h-\varepsilon) - \lim_{n\to\infty}\left(\frac{2^{n+1}-\Sigma_{i=0}^{m(n)}\sigma^{s_i}}{\sum_{i=0}^{m(n)}\sigma^{s_i}}\right)\log 2 \\
&= \mathbf{K}(S)(h-\varepsilon) - \mathbf{C}(S)\log 2.
\end{aligned}$$

This completes the proof. □

The following theorem gives that positive topological entropy implies positive h_{top}^S.

Theorem 2.3. *Let A be a primitive matrix and let n_A be a positive integer such that $A^{n_A} > 0$. If $k_n \geq n_A$ for all sufficiently large n, then*

$$h_{top}^S(\mathcal{T}_A) \geq h_{top}(\mathcal{T}_A).$$

Proof. We will construct a subset of $\mathcal{T}_A$ such that $h^S_{top}(\mathcal{T}_A) \geq \log s$ where s is the maximum row sum of A. Let $a \in \mathcal{A}$ be a symbol attaining s. Since $k_n \geq n_A$ for all $n \in \mathbb{Z}_+$, for each $b \in \mathcal{A}$ there is a path from b to a in exactly k_n steps. The tree t will be constructed in following steps, on row $s_0 - 1$ assign a at every vertex, then each vertex on the row s_0 has s choices. For each symbol on the row s_0, there is a path such that this symbol reach the symbol i on the row $s_1 - 1 = s_0 + k_0 - 1$. Then on the row s_1, each vertex has s choices. Repeating the same process on each $s_n - 1 = s_0 + \sum_{i=0}^{n-1} k_i - 1$ and $s_0 + \sum_{i=0}^{n-1} k_i = s_n$ rows, we have that every vertex on row s_n will have s choices. It is easy to check the tree $t \in \mathcal{T}_A$. Then

$$h^S_{top}(\mathcal{T}_A) \geq \lim_{n\to\infty} \frac{\log s^{\sum_{i=0}^{n-1} |\Sigma^{s_i}|}}{\sum_{i=0}^{n-1} |\Sigma^{s_i}|} = \log s.$$

The proof is complete due to the fact that $h_{top}(\mathcal{T}_A) \leq \log s$. □

We remark that there is an A such that $h^S_{top}(\mathcal{T}_A) > h_{top}(\mathcal{T}_A)$ (see Thoerem 3.5).

2.3. **Characterization of the null systems.** The following theorem provides an equivalence condition for hom tree-shifts.

Theorem 2.4. *Let $X \subseteq \mathcal{A}^{\mathbb{Z}_+}$ be a shift space and define*

$$r_n = \max_{w_n \in B_n(X)} |\{a \in \mathcal{A} : w_n a \in B_{n+1}(X)\}|.$$

We have $h^S_{top}(\mathcal{T}_X) > 0$ for all $S = \{s_n\}_{n=0}^\infty$ if and only if $r_n > 1$ for all $n \in \mathbb{N}$. Equivalently, this means X is a shift of finite type containing only finite elements.

Proof. Let $\mathcal{T}$ be a d-tree ($d \geq 2$). If $r_n = 1$ for all sufficiently large n, then $|\mathcal{T}_X|$ is finite. This implies $|B_n(\mathcal{T}^S_X)| \leq |\mathcal{T}_X|$ for all $n \in \mathbb{N}$ and for all $S = \{s_n\}_{n=0}^\infty$. Thus, $h^S_{top}(\mathcal{T}_X) = 0$. If $r_n > 1$ for all $n \in \mathbb{N}$, then for any $S = \{s_n\}_{n=0}^\infty$ there is a sequence of words $\{w_{s_n-1}\}_{n=0}^\infty$ such that

$$|\{a \in \mathcal{A} : w_{s_n-1} a \in B_{s_n}(X)\}| \geq 2.$$

This implies

$$\left|B_n(\mathcal{T}^S_X)\right| \geq 2^{d^{s_n}}.$$

Then we have

$$\frac{\log \left|B_n(\mathcal{T}^S_X)\right|}{\sum_{i=0}^n d^{s_i}} \geq \frac{d^{s_n} \log 2}{\sum_{i=0}^n d^{s_i}} \geq \frac{d-1}{d} \log 2 > 0$$

for all $n \in \mathbb{N}$. Thus, $h^S_{top}(\mathcal{T}_X) > 0$.

Now note that if X is a shift of finite type which contains only finite elements, then clearly $r_n \geq 1$ for all sufficiently large n. Conversely, if there exists $N \in \mathbb{N}$ such that $r_n = 1$ for $n \geq N$, then it is easy to see X can be determined by the forbidden set which consists of all words of length $N+1$ not appearing in X. Therefore, X is a shift of finite type and has finite members. □

3. The supremum of h^S_{top} over all subsequences of $\mathbb{N}$

3.1. **Topological surface entropy.** This subsection carries out the discussion of the relationship between h^S_{top} and topological surface entropy of a hom tree-shift $\mathcal{T}_X$. For convenience, we denote by L_n the sets Σ^n, over which the following two sets of patterns are defined:

$$B^L_n(\mathcal{T}_X) = \{t|_{L_n} : t \in \mathcal{T}_X\} \text{ and } B^L_{n;u}(\mathcal{T}_X) = \{t|_{L_n} : t \in \mathcal{T}_X, t|_{s(u)} = u\},$$

where $u : s(u) \to \mathcal{A}$ is a function on a finite subset $s(u)$ of $\mathcal{T}$. We simply write $B^L_{n;a}(\mathcal{T}_X)$ when $s(u) = \{\epsilon\}$ and $u(\epsilon) = a$. In the same vein of the topological entropy, the topological surface entropy is defined to be the growth rate of $B^L_n(\mathcal{T}_X)$ written as

$$h^L_{top}(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log \left|B^L_n(\mathcal{T}_X)\right|}{|L_n|}.$$

This growth rate is shown in the following proposition to exist in the decreasing manner.

Proposition 3.1. *Let $\mathcal{T}_X$ be a hom tree-shift. Then, $\frac{\log\left|B^L_n(\mathcal{T}_X)\right|}{|L_n|}$ is decreasing. If in addition that $\mathcal{T}_X = \mathcal{T}_A$ is a hom Markov tree-shift with a primitive adjacency matrix A, then both $\max_{a\in\mathcal{A}} \frac{\log\left|B^L_{n;a}(\mathcal{T}_X)\right|}{|L_n|}$ and $\min_{a\in\mathcal{A}} \frac{\log\left|B^L_{n;a}(\mathcal{T}_X)\right|}{|L_n|}$ increase to $h^L(\mathcal{T}_X)$.*

Proof. It follows from the property of shift invariance that

$$\left|B^L_{n+1}(\mathcal{T}_X)\right| \le \left|B^L_n(\mathcal{T}_X)\right|^d.$$

If, in addition, $\mathcal{T}_X = \mathcal{T}_A$ with A primitive, then by placing two elements of $B^L_{n;a}(\mathcal{T}_A)$ side by side, one obtains an element of $B^L_{n+1;a}(\mathcal{T}_A)$, and thus

$$\max_{a\in\mathcal{A}} \left|B^L_{n+1;a}(\mathcal{T}_A)\right| \ge \max_{a\in\mathcal{A}} \left|B^L_{n;a}(\mathcal{T}_A)\right|^d,$$

$$\min_{a\in\mathcal{A}} \left|B^L_{n+1;a}(\mathcal{T}_A)\right| \ge \min_{a\in\mathcal{A}} \left|B^L_{n;a}(\mathcal{T}_A)\right|^d.$$

Hence, monotonicity follows immediately. On the other hand, by assuming that A is primitive with $A^N > 0$, we have

$$|\mathcal{A}|^{-1} \left|B^L_n(\mathcal{T}_A)\right| \le \max_{a\in\mathcal{A}} \left|B^L_{n;a}(\mathcal{T}_A)\right| \le \left|B^L_n(\mathcal{T}_A)\right|,$$

$$\max_{a\in\mathcal{A}} \left|B^L_{n;a}(\mathcal{T}_A)\right|^{d^N} \le \min_{a\in\mathcal{A}} \left|B^L_{n+N;a}(\mathcal{T}_A)\right| \le \max_{a\in\mathcal{A}} \left|B^L_{n+N;a}(\mathcal{T}_A)\right|,$$

and thus the proposition holds. □

An immediate consequence of the proposition above is that the topological surface entropy is an upper bound for h^S_{top}, which is stated in the following proposition.

Proposition 3.2. *Let $\mathcal{T}_X$ be a hom tree-shift. If S is strictly increasing as assumed, then for all $n \ge 0$,*

$$\frac{\log\left|B_n(\mathcal{T}^S_X)\right|}{|\Delta^S_n|} \le \frac{\sum_{i=0}^n \log\left|B^L_{s_i}(\mathcal{T}_X)\right|}{\sum_{i=0}^n |L_{s_n}|}.$$

Therefore, $h^S_{top}(\mathcal{T}_X) \le h^L_{top}(\mathcal{T}_X)$.

Proof. The proposition follows as a result of

$$|B_n| \le \prod_{i=1}^n \left|B^L_{s_i}\right| \quad \text{and} \quad \left|\Delta^S_n\right| = |\cup_{i=1}^n L_{s_i}| = \sum_{i=1}^n |L_{s_i}|.$$

The proof is finished by taking the limit from both sides of the inequality. □

Corollary 3.3. *Let $\mathcal{T}_X$ be a hom tree-shift and S be strictly increasing. Then,*

$$h_{top}(\mathcal{T}_X) \le h^\infty_{top}(\mathcal{T}_X) = h^L_{top}(\mathcal{T}_X).$$

Proof. The idea behind the corollary is the same as that in [26].

It follows from the definition that $h_{top}(\mathcal{T}_X) \leq h^{\infty}_{top}(\mathcal{T}_X)$, so it is left to show the remaining equality. Since Proposition 3.2 indicates that $h^S_{top}(\mathcal{T}_X) \leq h^L_{top}(\mathcal{T}_X)$ for every strictly increasing S, the last step of the proof is to show this upper bound can be attained by any increasing sequence S with unbounded gaps. For this choice, there exists a subsequence s_{n_i} of s_n such that $\lim_{n\to\infty} \frac{|L_{s_{n_i}}|}{|\Delta^S_{n_i}|} = 1$. Since $\left|B^L_{s_{n_i}}(\mathcal{T}_X)\right| \leq \left|B_{n_i}(\mathcal{T}^S_X)\right|$, we have

$$
\begin{aligned}
&h^S_{top}(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log\left|B_n(\mathcal{T}^S_X)\right|}{|\Delta^S_n|} \\
&\geq \limsup_{i\to\infty} \frac{\log\left|B_{n_i}(\mathcal{T}^S_X)\right|}{\left|\Delta^S_{n_i}\right|} = \limsup_{i\to\infty} \frac{\log\left|B_{n_i}(\mathcal{T}^S_X)\right|}{\left|L_{s_{n_i}}\right|} \frac{|L_{n_i}|}{\left|\Delta^S_{n_i}\right|} \\
&\geq \lim_{n\to\infty} \frac{\log\left|B^L_{s_{n_i}}(\mathcal{T}_X)\right|}{\left|L_{s_{n_i}}\right|} = h^L_{top}(\mathcal{T}_X).
\end{aligned}
$$

This completes the proof. □

In the following, we are going to deduce the asymptotic behavior of topological surface entropy with respect to the number of generators d, and therefore we will denote by $\mathcal{T}^{(d)}$ the d-tree to put an emphasis on the number of the generators.

Remark 3.4. We claim that, for any hom tree-shift $\mathcal{T}_X$, the number $h^L_{top}(\mathcal{T}^{(d)}_X)$ approaches the limit $\lim_{n\to\infty} \log r_n$ (as $d \to \infty$), where r_n is defined in Theorem 2.4. To show this, it is noteworthy that

$$
\max_{u\in B_{n-1}(\mathcal{T}^{(d)}_X)} \frac{\log\left|B^L_{n;u}(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_n\right|} = \log r_n.
$$

is independent of dimension d. Since r_n is decreasing, we take n_0 such that $r_{n_0} = \lim_{n\to\infty} r_n$ and apply Proposition 3.1 and the pigeonhole principle to deduce that

$$
\begin{aligned}
&\log r_{n_0} + \frac{\log\left|B_{n_0-1}(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_{n_0}\right|} = \frac{\log\left(\left|B_{n_0-1}(\mathcal{T}^{(d)}_X)\right| \cdot \max_{u\in B_{n_0-1}(\mathcal{T}^{(d)}_X)} \left|B^L_{n_0;u}(\mathcal{T}^{(d)}_X)\right|\right)}{\left|L^{(d)}_{n_0}\right|} \\
&\geq \frac{\log\left|B^L_{n_0}(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_{n_0}\right|} \geq h^L_{top}(\mathcal{T}^{(d)}_X) = \inf_{n\to\infty} \frac{\log\left|B^L_n(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_n\right|} = \lim_{n\to\infty} \frac{\log\left|B^L_n(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_n\right|} \\
&\geq \lim_{n\to\infty} \log r_n = \lim_{n\to\infty} \max_{u\in B_{n-1}(\mathcal{T}^{(d)}_X)} \frac{\log\left|B^L_{n;u}(\mathcal{T}^{(d)}_X)\right|}{\left|L^{(d)}_n\right|}.
\end{aligned}
$$

The result then follows due to the fact $|\Delta_{n_0-1}| / \left|L^{(d)}_{n_0}\right|$ tends to zero. As a consequence, this limit $\lim_{n\to\infty} \log r_n$ is the logarithm of an integer. It is noteworthy that this result is also obtained for the irreducible hom Markov tree-shift in [26], and this limit is expressed therein as the logarithm of maximal row sum of the adjacency matrix.

3.2. **The supremum of h_{top}^{S} of all subsequence of $\mathbb{N}$.** We provide in the following an example whose maximal pattern entropy fails to be the logarithm of an integer.

Theorem 3.5. *There exists a hom Markov tree-shift $\mathcal{T}_A$ over 5-tree such that $h_{top}^{\infty}(\mathcal{T}_A) \in (\log 4, \log 5)$.*

Proof. Let $\mathcal{T}_A$ be a hom Markov tree-shift over 5-tree determined by the adjacency matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

We show that $\log 4 < h_{top}^{\infty}(\mathcal{T}_A) < \log 5$. We first prove that $h_{top}^{\infty}(\mathcal{T}_A) < \log 5$. An immediate consequence of the definition of A is that the set $B_1^L(\mathcal{T}_A)$ does not contain any surjective function and thus has no more than $5^5 - 5!$ elements. Proposition 3.1 and 3.2 then yields

$$h_{top}^{\infty}(\mathcal{T}_A) \leq \frac{\log \left|B_1^L(\mathcal{T}_A)\right|}{|L_1|} \leq \frac{\log(5^5 - 5!)}{5} < \log 5.$$

As for lower bound, since

$$B_{n;0}^L(\mathcal{T}_A) \supset (\Sigma^2)^{\{0,1,2,3\}} \cup (\Sigma^2)^{\{4\}},$$

$B_{n;0}^L(\mathcal{T}_A)$ has at least $4^{25} + 1$ elements. Applying Proposition 3.1 and 3.2 results in

$$h_{top}^{\infty}(\mathcal{T}_A) \geq \frac{\log \left|B_{2;0}^L(\mathcal{T}_A)\right|}{|L_2|} \geq \frac{\log(4^{25} + 1)}{25} > \log 4.$$

The proof is complete. □

Since the idea behind the above theorem forms the backbone of the second part of Theorem 2.1, we are at this point ready to give a proof.

Proof of the (2) of Theorem 2.1. Under the assumption A is primitive, we claim that $h^L(\mathcal{T}_A) = h(\mathcal{T}_A)$ if and only if A is an all-one matrix. Since sufficiency is clear, we shall only prove the converse by considering a contrapositive statement. Suppose $a \in \mathcal{A}$ is a symbol whose row sum M is the maximum among all rows. The discussion is divided into the following two cases: (1) $M = |\mathcal{A}|$, and (2) $M < |\mathcal{A}|$. In the first case, it is clear that $h^L(\mathcal{T}_A) = \log M$ and it follows from [4, Theorem 2.2] that $h(\mathcal{T}_A) < \log M$ since A is not an all-one matrix. For the second case, note that

$$\begin{aligned} \left|B_{n_A+1;a}^L(\mathcal{T}_A)\right| &\geq \left|\{u \in B_{n_A+1;a}^L(\mathcal{T}_A) : A(a, t_\omega) = 1, \forall \omega \in L_{n_A}\}\right| \\ &\quad + \left|\{u \in B_{n_A+1;a}^L(\mathcal{T}_A) : \exists \omega \in L_{n_A}, A(a, t_\omega) = 0\}\right| \\ &> M^{\left|L_{n_A+1}\right|} \end{aligned}$$

since A is primitive and the latter term in the second line is clearly nonzero. Therefore, $h^L(\mathcal{T}_A) \geq \frac{\log\left|B_{n_A+1;a}^L(\mathcal{T}_A)\right|}{\left|L_{n_A+1}\right|} > \log M$ while $h(\mathcal{T}_A) \leq \log M$. To prove the theorem, it is sufficient to take a sequence S that contains arbitrarily long intervals while but that the gaps k_n is an unbounded sequence. It is then clear that

$$\limsup_{n\to\infty}\frac{\log\left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|}=h^L(\mathcal{T}_A),$$
$$\liminf_{n\to\infty}\frac{\log\left|B_n(\mathcal{T}_A^S)\right|}{|\Delta_n^S|}\leq h(\mathcal{T}_A)<h^L(\mathcal{T}_A).$$

□

3.3. **The denseness of h_{top}^S.** In this section, we show that given any $d\geq 2$, the set of all possible values of h_{top}^S.

$$\mathcal{H}_L^{(d)}=\left\{h_{top}^L(\mathcal{T}_A):\mathcal{T}_A \text{ is on } d\text{-tree and } A \text{ is binary and irreducible}\right\} \tag{17}$$

is dense in $[d\log 2,\infty)$.

Theorem 3.6. *For $d\geq 2$, $\mathcal{H}_L^{(d)}$ is dense in $[d\log 2,\infty)$.*

Before proving above theorem, we need some necessary notations. For $\ell\geq 1$, let $\mathbf{y}_\ell=\{y_i\}_{i=1}^{\ell}$ be a sequence of positive integers. Given $\mathbf{y}_\ell$ and $N\geq 1$, an irreducible directed graph $G=G_{\mathbf{y}_\ell;N}$ can be constructed as follows. The vertex set $V(G_{\mathbf{y}_\ell;N})$ of $G_{\mathbf{y}_\ell;N}$ is defined by

$$V(G_{\mathbf{y}_\ell;N})=\bigcup_{i=1}^{N+\ell}V_i=\bigcup_{i=1}^{N+\ell}\left\{v_{i;j}:1\leq j\leq n_i\right\},$$

where

$$n_i=\begin{cases}1 & \text{for } 1\leq i\leq N,\\ y_{N+\ell+1-i}\cdot y_{N+\ell-i}\cdot\dots\cdot y_\ell & \text{for } N+1\leq i\leq N+\ell.\end{cases} \tag{18}$$

For $N+1\leq i\leq N+\ell$ and $1\leq j\leq y_{N+\ell+1-i}$, we define

$$V_{i;j}=\left\{v_{i;k}:\frac{n_i(j-1)}{y_{N+\ell+1-i}}+1\leq k\leq\frac{n_ij}{y_{N+\ell+1-i}}\right\}.$$

Then, the edge set $E(G_{\mathbf{y}_\ell;N})$ of $G_{\mathbf{y}_\ell;N}$ is defined by

$$\begin{aligned}E(G_{\mathbf{y}_\ell;N})=&\left(\bigcup_{i=1}^{N}\left\{e(v_\alpha,v_\beta):v_\alpha\in V_i \text{ and }:v_\beta\in V_{i+1}\right\}\right)\\ &\bigcup\left(\bigcup_{i=N+1}^{N+\ell-1}\left\{e(v_{i;j},v_\alpha):v_\alpha\in V_{i+1;j}\right\}\right)\\ &\bigcup\left\{e(v_\alpha,v_{1;1}):v_\alpha\in V_{N+\ell}\right\},\end{aligned}$$

where $e(v_\alpha,v_\beta)$ means an edge from v_α to v_β. Clearly, $G_{\mathbf{y}_\ell;N}$ is an irreducible graph.

Given $n\geq 1$, for any finite sequence $\mathbf{b}_n=(b_1,b_2,\cdots,b_n)$, we define the (left) cyclic shift map $\sigma(\mathbf{b}_n)=(b_2,b_3,\cdots,b_n,b_1)$. Denote by $\mathcal{S}(\mathbf{b}_n)=\{\sigma^m(\mathbf{b}_n):1\leq m\leq n\}$.

Proof of Theorem 3.6. Given a sequence of positive integers $\mathbf{y}_\ell=\{y_i\}_{i=1}^{\ell}$, $\ell\geq 1$, and $N\geq 1$, the irreducible graph $G=G_{\mathbf{y}_\ell;N}$ can be constructed as above. First, for $n\geq\ell$, by considering the case that the symbols on L_{n-i} belong to $V_{N+\ell-i}$, $0\leq i\leq\ell-1$, it can be verified that

$$B_n^L(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})\geq y_1^{d^n}\cdot y_2^{d^{n-1}}\cdot\dots\cdot y_\ell^{d^{n-\ell+1}}, \tag{19}$$

which implies that

$$\begin{aligned}
h^L(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}) \geq & \lim_{n\to\infty} \tfrac{\log y_1^{d^n}\cdot y_2^{d^{n-1}}\cdot \ldots \cdot y_\ell^{d^{n-\ell+1}}}{d^n} \\
= & \log y_1 + \tfrac{1}{d}\log y_2 + \cdots + \tfrac{1}{d^{\ell-1}}\log y_\ell .
\end{aligned} \tag{20}$$

Secondly, it is clear that $B_n^L(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}) \leq B_n(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})$. We then have that

$$\begin{aligned}
h_{top}^L(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}) \leq & \limsup_{n\to\infty} \tfrac{\log B_n(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})}{|L_n|} \\
= & \limsup_{n\to\infty} \tfrac{\log B_n(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})}{|\Delta_n|} \cdot \limsup_{n\to\infty} \tfrac{|\Delta_n|}{|L_n|} \\
= & \tfrac{d}{d-1} h_{top}(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}).
\end{aligned} \tag{21}$$

For convenience, $v_{1;1}$ is labeled by 1. Let

$$m_i = \begin{cases} 1 & 1 \leq i \leq N \\ y_{n+\ell+1-i} & N+1 \leq i \leq N+\ell. \end{cases} \tag{22}$$

For $i \geq N+l+1$, $m_i = m_r$ where $i \equiv r (\mathrm{mod}\ N+l)$ and $n_0 = n_{N+\ell}$. Then, by the construction of G, it is easy to verified that

$$B_{n;1}(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}) = \prod_{i=0}^{n} m_{i+1}^{d^i}.$$

From [25, Theorem 3.3], since G is irreducible,

$$h_{top}\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) = \limsup_{n\to\infty} \frac{\log B_{n;1}(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})}{|\Delta_n|}.$$

Let $\mathbf{z}_{N+\ell} = (m_{N+\ell}, m_{N+\ell-1}, \cdots, m_1)$ be the finite sequence. By (22),

$$\begin{aligned}
h_{top}\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) = & \limsup_{n\to\infty} \tfrac{\log B_{n;1}(\mathcal{T}_{G_{\mathbf{y}_\ell;N}})}{|\Delta_n|} \\
= & \limsup_{n\to\infty} \tfrac{1}{|\Delta_n|} \sum_{i=0}^{n} d^i \log n_{i+1} \\
= & \tfrac{d^{N+\ell}(d-1)}{d^{N+\ell}-1} \left(\max_{\mathbf{w}_{N+\ell}\in\mathcal{S}(\mathbf{z}_{N+\ell})} \left\{ \tfrac{1}{d}\log w_1 + \tfrac{1}{d^2}\log w_2 + \cdots + \tfrac{1}{d^{N+\ell}}\log w_{N+\ell} \right\} \right) \\
= & \tfrac{d^{N+\ell}(d-1)}{d^{N+\ell}-1} \left(\max_{\mathbf{w}_\ell\in\mathcal{S}(\mathbf{y}_\ell)} \left\{ \tfrac{1}{d}\log w_1 + \tfrac{1}{d^2}\log w_2 + \cdots + \tfrac{1}{d^\ell}\log w_\ell \right\} \right).
\end{aligned}$$

Hence,

$$\begin{aligned}
& \lim_{N\to\infty} h_{top}\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) \\
= & (d-1) \max_{\mathbf{w}_\ell\in\mathcal{S}(\mathbf{y}_\ell)} \left\{ \tfrac{1}{d}\log w_1 + \tfrac{1}{d^2}\log w_2 + \cdots + \tfrac{1}{d^\ell}\log w_\ell \right\} \\
= & (d-1) \max_{0\leq m\leq \ell-1} \left\{ \tfrac{1}{d}\log y_{m+1} + \cdots + \tfrac{1}{d^{\ell-m}}\log y_\ell + \tfrac{1}{d^{\ell-m+1}}\log y_1 + \cdots + \tfrac{1}{d^\ell}\log y_m \right\}.
\end{aligned}$$

In particular, consider $\mathbf{y}_\ell = \{y_i\}_{i=1}^{\ell}$ with $y_1 = 2^a$, $a \geq 1$, and $y_i \in \{2^j : 0 \leq j \leq d-1\}$ for $2 \leq i \leq \ell$. From direct computation, we can conclude that

$$\lim_{N\to\infty} h_{top}\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) = (d-1)\left(\tfrac{1}{d}\log y_1 + \tfrac{1}{d^2}\log y_2 + \cdots + \tfrac{1}{d^\ell}\log y_\ell\right)$$

$$\Leftrightarrow \quad \left(\tfrac{1}{d} - \tfrac{1}{d^{\ell-m+1}}\right)\log y_1 \geq \sum_{i=1}^{\ell-m}\left(\tfrac{1}{d^i} - \tfrac{1}{d^{m+i}}\right)\log y_{m+i} - \sum_{i=2}^{m}\left(\tfrac{1}{d^i} - \tfrac{1}{d^{i+\ell-m}}\right)\log y_i$$

$$\Leftrightarrow \quad \left(\tfrac{1}{d} - \tfrac{1}{d^{\ell-m+1}}\right)\log y_1 \geq \tfrac{d^2}{d-1}\left(\tfrac{1}{d} - \tfrac{1}{d^{m+1}}\right)\left(\tfrac{1}{d} - \tfrac{1}{d^{\ell-m+1}}\right)\log 2^{d-1}$$

$$\Leftrightarrow \quad \log y_1 \geq \tfrac{d^2}{d-1}\left(\tfrac{1}{d} - \tfrac{1}{d^{m+1}}\right)\log 2^{d-1}$$

$$\Leftrightarrow \quad a \geq d^2\left(\tfrac{1}{d} - \tfrac{1}{d^{m+1}}\right).$$

for $1 \leq m \leq \ell - 1$. Hence, if $a \geq d$, then

$$\lim_{N\to\infty} h_{top}\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) = (d-1)\left(\frac{1}{d}\log y_1 + \frac{1}{d^2}\log y_2 + \cdots + \frac{1}{d^\ell}\log y_\ell\right).$$

for all $\ell \geq 1$. Combining with (20) and (21) yields that

$$\lim_{N\to\infty} h^L_{top}(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}) = \log y_1 + \frac{1}{d}\log y_2 + \cdots + \frac{1}{d^{\ell-1}}\log y_\ell. \tag{23}$$

It can be easily checked that

$$\left\{\lim_{N\to\infty} h^L\left(\mathcal{T}_{G_{\mathbf{y}_\ell;N}}\right) : y_1 \geq 2^d \text{ and } y_i \in \{2^j : 0 \leq j \leq d-1\} \text{ for } 2 \leq i \leq \ell, \ell \geq 1\right\}$$

is dense in

$$[\log y_1, \log y_1 + \log 2).$$

Clearly,

$$\log(y_1 + 1) \leq \log y_1 + \log 2$$

for $y_1 \geq 2^d$. Therefore, $\mathcal{H}_L^{(d)}$ is dense in

$$\bigcup_{y_1 \geq 2^d} [\log y_1, \log y_1 + \log 2) = [d\log 2, \infty).$$

The proof is completed. □

4. The induced entropy

Let $S \subseteq \mathbb{Z}_+$ and $K = \{k_i\}_{i=0}^{\infty}$ be defined as before. If $s_0 = 0$ and $k_n = 1$ for all $n \geq 0$, that is $S = \{i\}_{i=0}^{\infty}$, we have $h^S_{top}(\mathcal{T}_A) = h_{top}(\mathcal{T}_A)$ clearly. Another concept, namely $\mathcal{T}_A^{S,ind}$, is introduced in (10) and the *topological sequence entropy* of $h^{S;(i)}_{top}(\mathcal{T}_A)$ is similarly defined.

Since $\mathcal{T}_A^{S,ind}$ or $\mathcal{T}_A^S$ can be seen as a sort of NDS, this study could be useful for further development of the entropy theory for symbolic dynamical systems on tree (or countable graphs), results in this area is sparse, some results on Markov chain can be found in [29, 10, 9]. Furthermore, as we described above, the sequence entropy has a close relationship to the mixing properties of a dynamical system, which is an interesting direction for future study.

4.1. **Characterization of the null systems.**

Theorem 4.1. *Let $X_A \subseteq \mathcal{A}^{\mathbb{Z}_+}$ be a Markov shift space and*

$$r_n = \max_{w_n \in X_n} \left|\{a \in \mathcal{A} : w_n a \in B_{n+1}(X_A)\}\right|.$$

We have $h_{top}^{S;(i)}(\mathcal{T}_X) > 0$ for all $K = \{k_n\}_{n=0}^{\infty}$ if and only if $r_n > 1$ for all $n \in \mathbb{N}$.

Proof. If $r_n = 1$ for all sufficiently large n, then $|\mathcal{T}_A|$ is finite. This implies $|B_n(\mathcal{T}_A^S)| \leq |\mathcal{T}_A|$ for all $n \in \mathbb{N}$ and for all $S = \{s_n\}_{n=0}^{\infty}$. Thus, $h_{top}^S(\mathcal{T}_A) = 0$. If $r_n > 1$ for all $n \in \mathbb{N}$, then for any $S = \{s_n\}_{n=0}^{\infty}$ there is a sequence of words $\{w_{n-1}\}_{n=0}^{\infty}$ such that

$$|\{a \in \mathcal{A} : w_{n-1} a \in B_n(X_A)\}| \geq 2.$$

This implies

$$\left|B_n(\mathcal{T}_X^{S,ind})\right| \geq 2^{d^n}.$$

Then we have

$$\frac{\log \left|B_n(\mathcal{T}_X^{S,ind})\right|}{\sum_{i=0}^n d^i} \geq \frac{d^n \log 2}{\sum_{i=0}^n d^i} \geq \frac{d-1}{d} \log 2 > 0$$

for all $n \in \mathbb{N}$. Thus, $h_{top}^{S;(i)}(\mathcal{T}_X) > 0$. □

4.2. **Comparison betweem $h_{top}^{S;(i)}(\mathcal{T}_A)$, $h_{top}^S(\mathcal{T}_A)$ and $h_{top}(\mathcal{T}_A)$.**

Theorem 4.2. *If $k_n \geq n_A$ for all $n \geq M$, then*

$$h_{top}^{S;(i)}(\mathcal{T}_A) \geq h_{top}^S(\mathcal{T}_A) \geq h_{top}(\mathcal{T}_A).$$

Proof. It is clear that $h_{top}^{S;(i)}(\mathcal{T}_A) = \log m$, then the proof is complete by the Theorem 2.3 and the fact that $h_{top}^S(\mathcal{T}_A) \leq \log m$. □

The following example obtain the strictly case in Theorem 4.2.

Example 4.3 **($h_{top}^{S;(i)}(\mathcal{T}_A) > h_{top}^S(\mathcal{T}_A) > h_{top}(\mathcal{T}_A)$).** Let $s_0 = 0$ and $k_n = 4, \forall n \in \mathbb{Z}_+$ and $\mathcal{T}_A$ be the 2-tree with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then $h_{top}^S(\mathcal{T}_A) \geq \log 2$ by the proof of Theorem 4.2 (2), and we also have $\log 2 > h_{top}(\mathcal{T}_A)$. In fact, since A^4 is a positive matrix, we have $h_{top}^{S;(i)}(\mathcal{T}_A) = \log 3$. On the other hand, Proposition 3.1 and 3.2 imply that

$$\log 3 > \frac{\log 5}{2} = \frac{\log \left|B_1^L(\mathcal{T}_X^S)\right|}{2} \geq h_{top}^L(\mathcal{T}^S) \geq h_{top}^S(\mathcal{T}_A).$$

Acknowledgement

We would like to sincerely thank the anonymous referee for providing helpful suggestions for the first draft of this article. These improve the readability of the paper.

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