TOPOLOGICAL ENTROPY AND SEQUENCE ENTROPY FOR HOM TREE-SHIFTS ON UNEXPANDABLE TREES

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Abstract. This article explores the topological entropy and topological sequence entropy of hom tree-shifts on unexpandable trees. Regarding topological entropy, we establish that the entropy, denoted as $h(\mathcal{T}_X)$ on an unexpandable tree, equals the entropy $h(X)$ of the base shift X when X is a subshift satisfying the almost specification property. Additionally, we derive some fundamental properties such as entropy approximation and the denseness of entropy for subsystems. Concerning topological sequence entropy, we show that the set of sequence entropies of hom tree-shifts with a base shift is generated by an irreducible matrix A, forming a subset of $\log \mathbb{N}$. Precisely, these entropies correspond to the logarithms of the largest cardinalities of the periodic classes of A.

Contents

1. Introduction

Let $\mathcal{A}$ be a finite set with $|\mathcal{A}| \geq 2$, and let T be a *tree*, which is a countable graph that is locally finite, without loops, and has a root ϵ. A tree T can also be defined

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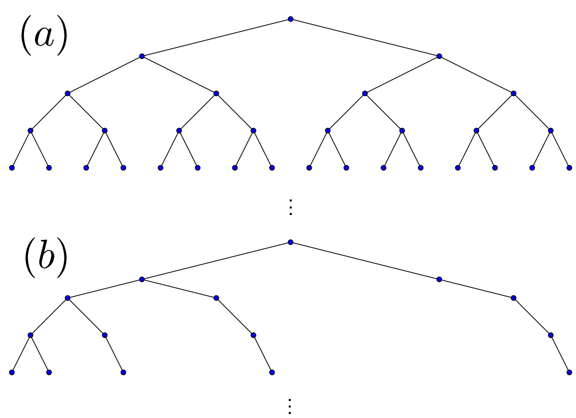

FIGURE 1. (a) The figure of T^2 and (b) the figure of T^M.

as the Cayley graph of the semigroup with d generators, denoted as $g_1, \dots, g_d$. Let $M \in \{0,1\}^{d\times d}$, and the associated *Markov-Cayley tree* T^M is defined as follows:

$$T^M = \{\epsilon\} \cup \{g_{i_1} \cdots g_{i_n} : n \geq 1,\ M(i_j, i_{j+1}) = 1\ \forall j = 1, \dots, n-1\}.$$

For $i \in \mathbb{N} \cup \{0\}$, we denote by T_i the set of vertices in T with length i, where the *length* of a vertex $g \in T$ is the number of edges from ϵ to g. For any tree T, we associate the number

$$\gamma_T := \lim_{n\to\infty} \frac{|T_{n+1}|}{|T_n|} \geq 1,$$

which is called the *expanding number* of the tree T whenever the limit exists. We call T *expandable* if $\gamma_T > 1$ and *unexpandable* if $\gamma = 1$. It is worth noting that the conventional d-tree T^d, the free semigroup with d generators (see Figure 1 (a) for T^2) is expandable. Furthermore, the tree $T = T^M$ with $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is an unexpandable tree (see Figure 1 (b) for T^M).

Let $X \subseteq \mathcal{A}^{\mathbb{N}}$ be an $\mathbb{N}$ shift. The *tree-shift associated with* X (or *hom tree-shift with base shift* X, *tree-shift* for short) is defined as

$$\mathcal{T}_X = \{x \in \mathcal{A}^T :\ (x_{g_{i_1} g_{i_2} \cdots g_{i_j}})_{j \in \mathbb{N}\cup\{0\}} \in X \text{ for any } (g_{i_1} g_{i_2} \cdots g_{i_j})_{j\in\mathbb{N}\cup\{0\}}\},$$

where the subshift X is called the *base shift* of $\mathcal{T}_X$ [36]. The term 'hom' indicates that the rule X on every infinite path (starting from ϵ) of the tree T is the same. Suppose the base shift $X = X_A$ is a subshift of finite type (SFT) with adjacency matrix A, then we simply write $\mathcal{T}_A := \mathcal{T}_{X_A}$, and call it a *tree-SFT*. The tree-shift $\mathcal{T}_A$ can also be characterized as below.

$$\mathcal{T}_A = \{x \in \mathcal{A}^T : A_{x(g),x(h)} = 1\ \forall g, h \in T \text{ and } h = g g_j \text{ for } 1 \leq j \leq d\}.$$

The tree-shift $\mathcal{T}_X$ has received extensive attention in recent years [1, 2, 14, 35, 36, 5, 4] for the following three reasons: **(a)** such a shift has an intermediate class of symbolic dynamics between $\mathbb{N}$ shifts and $\mathbb{N}^d$ shifts [1]; **(b)** $\mathcal{T}_X$ exhibits very interesting phenomena which are different from $\mathbb{N}^d$ shifts for $d \geq 2$ [7] since the amenability[1] is no longer true for T; **(c)** $\mathcal{T}_X$ is a type of physical model in statistical physics that has attracted a lot of attention over the past two decades, e.g., regarding the large deviation principle [15, 16, 17], and the existence of the Gibbs states [39, 21].

Let $X \subseteq \mathcal{A}^{\mathbb{N}}$ and $\mathcal{T}_X$ be the associated tree-shift, the *entropy* of $\mathcal{T}_X$ is defined as

$$h(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log |\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|}, \tag{1}$$

where $\Delta_n = \cup_{i=0}^{n} T_i$ and $\mathcal{P}(\Delta_n, \mathcal{T}_X) : \mathcal{A}^T \to \mathcal{A}^{\Delta_n}$ is the *canonical projection* of $\mathcal{T}_X$ into the subtree Δ_n. Precisely,

$$\mathcal{P}(\Delta_n, \mathcal{T}_X) = \left\{ (x_g)_{g\in\Delta_n} \in \mathcal{A}^{\Delta_n} : \ x \in \mathcal{T}_X \right\}.$$

The limit (1) exists for tree-shifts defined on $T = T^2$ [35], and also exists for a larger class of tree-shifts defined on Markov-Cayley trees [9]. The main purpose of this article is to establish the theory of entropy for tree-shifts defined on **unexpandable trees**. Furthermore, we provide a comparison of entropies for tree-shifts defined on both expandable and unexpandable trees. Note that most of all results in this paper do not apply to expandable trees. In what follows, we assume that the tree T has no leaves. Equivalently, this means that each ray from the root ϵ in T has infinite cardinality. Below is a summary of our findings from this investigation.

1.1. Relations of entropy between tree-shifts and base shifts.

Suppose $T = T^2$, Petersen and Salama [35] prove that $h(\mathcal{T}_X) \geq h(X)$ if X is an irreducible SFT ([35, Theorem 3.3]), and Ban et al. [8] prove that under the same assumptions as [35, Theorem 3.3], we have $h(\mathcal{T}_X) = h(X)$ if and only if the adjacency matrix A of X has equal row sums; that is, $\max_i \sum_j A_{ij} = \min_j \sum_j A_{ij}$. For tree-shifts defined on an unexpandable tree T, the following two results provide sufficient conditions to ensure when the entropies for $h(\mathcal{T}_X)$ and $h(X)$ coincide.

Theorem 1.1. *Suppose T is an unexpandable tree and $\lim_{n\to\infty} |T_n| < \infty$. If $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a subshift, then*

$$h(\mathcal{T}_X) = h(X). \tag{2}$$

Recall that a subshift $X \subseteq \mathcal{A}^{\mathbb{N}}$ satisfies the *almost specification property* if

$$\exists c > 0, \forall u, v \in \mathcal{L}(X), \exists w \in \mathcal{L}(X) \text{ with } |w| \leq c \text{ such that } uwv \in \mathcal{L}(X),$$

where $\mathcal{L}(X) = \cup_{n=1}^{\infty} \mathcal{P}([1, n], X)$ and $\mathcal{P}([1, n], X) = \{(x_i)_{i=1}^n : x \in X\}$. For a general unexpandable tree T, we have the following results.

Theorem 1.2. *Suppose T is an unexpandable tree. Then for any subshift $X \subseteq \mathcal{A}^{\mathbb{N}}$, we have $h(\mathcal{T}_X) \leq h(X)$. Furthermore, if $X \subseteq \mathcal{A}^{\mathbb{N}}$ is a subshift satisfying the almost specification property, then the equality (2) holds true.*

Let $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $X = X_G$ be the golden-mean shift. Then by Theorem 1.1, we have $h(\mathcal{T}_X^M) = \log \frac{1+\sqrt{5}}{2} = h(X)$. If $M = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $h(\mathcal{T}_X^M) = \log \frac{1+\sqrt{5}}{2} = h(X)$ by Corollary 2.1.

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[1]That is, $|\Delta_n \backslash \Delta_{n-1}| / |\Delta_n|$ does not tend to 0 as $n \to \infty$, where $\Delta_n = \cup_{i=0}^{n-1} T_i$. For a deeper discussion of the amenability of a group we refer the reader to [14].

We suspect that the equality (2) holds true for any tree-shift defined on an unexpandable tree. However, our approach does not work under these circumstances. We list it as an open problem.

Problem 1. Let T be an unexpandable tree. For any tree-shift $\mathcal{T}_X$ defined on T, we have $h(\mathcal{T}_X) = h(X)$.

1.2. **Fundamental properties of entropy.** Suppose $T = T^2$ and $\mathcal{T}_A$ is a tree-SFT defined on T. For $a \in \mathcal{A}$, we define

$$\mathcal{P}_a(\Delta_n, \mathcal{T}_X) = \left\{(x_g)_{g\in\Delta_n} \in \mathcal{A}^{\Delta_n} : x \in \mathcal{T}_X \text{ and } x_\epsilon = a\right\},$$

and denote

$$h_a(\mathcal{T}_A) := \limsup_{n\to\infty} \frac{\log |\mathcal{P}_a(\Delta_n, \mathcal{T}_A)|}{|\Delta_n|}.$$

Petersen and Salama ([35, Proposition 3.1]) prove that if A is an irreducible 0-1 matrix, then for all $a, b \in \mathcal{A}$, we have

$$h_a(\mathcal{T}_A) = h_b(\mathcal{T}_A) = h(\mathcal{T}_A).$$

However, the authors also show that ([35, Example 3.2]) there exists an irreducible A such that

$$\liminf_{n\to\infty} \frac{\log |\mathcal{P}_a(\Delta_n, \mathcal{T}_A)|}{|\Delta_n|} < h_a(\mathcal{T}_A) = h(\mathcal{T}_A) \text{ for all } a \in \mathcal{A}.$$

That is, the limit $\frac{\log|\mathcal{P}_a(\Delta_n,\mathcal{T}_A)|}{|\Delta_n|}$ does not generally exist in this circumstance. Nevertheless, if A is primitive, then the limit in the definition of $h_a(\mathcal{T}_A)$ exists and

$$h_a(\mathcal{T}_A) = h(\mathcal{T}_A).$$

Suppose T is unexpandable; we prove that for any irreducible SFT X, and for all $a \in \mathcal{A}$, the limit

$$\lim_{n\to\infty} \frac{\log |\mathcal{P}_a(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|}$$

exists and is equal to $h(\mathcal{T}_X)$ (Theorem 1.3 (1)). This result is new and quite different from the case where T is expandable.

On the other hand, suppose A is a reducible matrix with irreducible components $A_1, A_2, \ldots, A_q$. The equality (3) is a well-known result for an $\mathbb{N}$ SFT ([33, Theorem 4.4.4])

$$h(X_A) = \max_{1\le i\le q} h(X_{A_i}), \tag{3}$$

where X_B is the $\mathbb{N}$ SFT and B is the corresponding adjacency matrix. For $T = T^2$, Ban et al. ([7, Theorem 3.2]) reveal that equality (3) is not generally true, i.e., $h(\mathcal{T}_A) \neq \max_{1\le i\le q} h(\mathcal{T}_{A_i})$. However, Theorem 1.3 (2) below shows that (3) holds for every tree-SFT defined on the unexpandable tree T.

Theorem 1.3. *Suppose T is an unexpandable tree, we have the following:*

1. *If X is an irreducible SFT, then we have the limit in the definition of $h_a(\mathcal{T}_X)$ exists and is equal to $h(\mathcal{T}_X)$ for all $a \in \mathcal{A}$.*
2. *Equality (3) holds for tree-SFTs defined on T. That is,*

$$h(\mathcal{T}_A) = \max_{1\le i\le q} h(\mathcal{T}_{A_i}).$$

1.3. **Possible values and denseness of entropy of the subsystems.** Understanding the possible values of the entropy for $\mathbb{N}$ SFTs or $\mathbb{N}^d$ SFTs is a critical and significant problem in the fields of dynamical systems and statistical physics, e.g., the hard-square model [11] and the ice model in $\mathbb{N}^2$ [30].

For $\mathbb{N}$ SFTs, it is known that the set $\{h(X) : X \subseteq \mathcal{A}^{\mathbb{N}}$ is a mixing SFT$\}$ is the logarithm of the numbers in the spectral radii of aperiodic non-negative integer matrices [31], and the set $\{h(X) : X \subseteq \mathcal{A}^{\mathbb{N}^d}$ is an SFT$\}$ ($d \geq 2$) is the class of non-negative right recursively enumerable numbers [23]. Desai [18] proved that any $\mathbb{N}^d$ SFT (resp. sofic) X with $h(X) > 0$ contains a family of $\mathbb{N}^d$ subSFTs (resp. subsofics) with entropies dense in the interval $[0, h(X)]$. Such a result sharpened an earlier result of Quas and Trow [37]. Combining the above facts implies that the set of possible entropies of $\mathbb{N}^d$ SFTs (or sofic shifts) is dense in $[0, \infty)$. Recently, Bland et al. [12] proved that for any countable amenable group G, if X is a G-SFT with positive topological entropy $h(X) > 0$, then the SFT subsystems of X are dense in the interval $[0, h(X)]$.

However, for $T = T^2$, Ban et al. ([10, Theorem 2.1]) proved that

$$\overline{\{h(\mathcal{T}_X) : X \text{ is an SFT and } \mathcal{T}_X \text{ is defined on } T\}} \cap (0, \frac{\log 2}{2}) = \emptyset,$$

where $\overline{S}$ stands for the closure of the set S. This means the set of possible values of the set $\{h(\mathcal{T}_X) : X$ is a SFT$\}$ is constrained. For tree-SFTs defined on unexpandable trees, we have the following result.

Theorem 1.4. *Under the same assumptions of Theorem 1.2, we have the following:*

1. *Suppose $\mathcal{T}_X$ is a tree-SFT with $h(\mathcal{T}_X) > 0$, then the set of entropies of tree-subSFTs of $\mathcal{T}_X$ is dense in $[0, h(\mathcal{T}_X)]$;*
2. *the following equality holds true.*

$$\overline{\{h(\mathcal{T}_X) : X \text{ is an SFT}\}} = [0, \infty).$$

Theorem 1.4 demonstrates that the structure of entropies of tree-shifts defined on unexpandable trees is similar to the one-dimensional case, as we mentioned in the preceding paragraph.

1.4. **Perturbations of the entropy.** Let $X \subseteq \mathcal{A}^{\mathbb{Z}}$ be a $\mathbb{Z}$ SFT, and w an admissible finite block. We denote by X_w the new SFT by adding w to the forbidden set. Lind [32] proves that for an irreducible $\mathbb{N}$ SFT X with $h(X) > 0$, there exist $C_X, D_X > 0$ and $N_X \in \mathbb{N}$ such that for any $n \geq N_X$ and any admissible block $w \in \mathcal{A}^{[1,n]}$,

$$\frac{C_X}{\exp(h(X)n)} < h(X) - h(X_w) < \frac{D_X}{\exp(h(X)n)}.$$

Pavlov ([34, Theorem 1.2]) extends the above result to $\mathbb{Z}^d$ SFTs for $d \geq 2$. He proves that for any strongly irreducible $\mathbb{Z}^d$ SFT X with $d \geq 2$ and $|X| > 1$, there exist C_X, $D_X > 0$, A_X, B_X, and $N_X \in \mathbb{N}$ such that for any $n > N_X$ and any admissible $w \in \mathcal{A}^{[1,n]^d}$, we have

$$\frac{C_X}{\exp(h(X)\,(n + A_X)^d)} < h(X) - h(X_w) < \frac{D_X}{\exp(h(X)\,(n + B_X)^d)}.$$

For a tree-SFT defined on an unexpandable tree, we have the following result.

Theorem 1.5. *Suppose T is an unexpandable tree and $\mathcal{T}_A$ is a tree-SFT with irreducible A. If $h(X_A) > 0$, there exist C_X, $D_X > 0$ and $N_X \in \mathbb{N}$ such that for any admissible block $w \in \mathcal{A}^{[1,n]}$ in X, we have*

$$\frac{C_X}{\exp(h(\mathcal{T}_X)n)} \leq h(\mathcal{T}_X) - h(\mathcal{T}_{X_w}) \leq \frac{D_X}{\exp(h(\mathcal{T}_X)n)}.$$

1.5. **Topological sequence entropy.** The notion of *sequence entropy* was first introduced by Kushnirenko [29], and is invariantly useful to characterize the mixing properties of measurable dynamical systems [25, 20, 38, 13, 3, 22, 24]. Suppose $f \in C(X)$ and X is a compact metric space. Let $S = \{s_n\}_{n=1}^{\infty}$ be a sequence of natural numbers, and α an open cover of X. Define $f^{-1}\alpha = \{f^{-1}A : A \in \alpha\}$, and define

$$h_{top}^S(f, \alpha) = \limsup_{n\to\infty} \frac{1}{n} \log N(f^{-s_1}\alpha \vee \cdots \vee f^{-s_n}\alpha),$$

where $N(\alpha)$ is the minimum cardinality of a subcover of α and $\alpha \vee \beta = \{A \cap B : A \in \alpha \text{ and } B \in \beta\}$. The *topological sequence entropy* of f with respect to $S = \{s_n\}_{n=1}^{\infty}$ is defined by

$$h_{top}^S(f) = \sup_{\alpha \in \mathcal{OC}(X)} h_{top}^S(f, \alpha),$$

where $\mathcal{OC}(X)$ is the collection of all finite open covers of X. Let $h_{top}^{\infty}(f)$[2] be the supremum of the topological sequence entropies of f over all subsequences of $\mathbb{N}$, and $\mathcal{H}^{\infty}(X) = \{h_{top}^{\infty}(f) : f \in C(X)\}$. It is known [26, 27] that

$$\mathcal{H}^{\infty}(X) \subseteq \{\infty, 0, \log 2, \log 3, \ldots\}. \tag{4}$$

If X is a finite tree or the unit circle S^1, then $\mathcal{H}^{\infty}(X) = \{\infty, 0, \log 2\}$. Related results can also be found in [41, 19, 40, 28]. It is interesting to note that the preceding setup and results are applied to an $\mathbb{N}$ shift without any difficulty.

Regarding tree-shifts defined on $T = T^2$, Ban et al. [6] introduce an analogous definition for topological sequence entropy using the concept of the *complete prefix code*, as detailed in Section 2. Suppose $\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty}$ is a collection of CPCs[3] and $\mathcal{T}_X \subseteq \mathcal{A}^T$ is a tree-shift, we define the *topological sequence entropy* of $\mathcal{T}_X$ with respect to $\mathcal{C}$ (written as $h_{top}^{\mathcal{C}}(\mathcal{T}_X)$) as

$$h_{top}^{\mathcal{C}}(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log \left|\mathcal{P}(\Delta_n^{\mathcal{C}}, \mathcal{T}_X)\right|}{|\Delta_n^{\mathcal{C}}|}, \tag{5}$$

where $\Delta_n^{\mathcal{C}} := \cup_{i=0}^{n-1}\mathcal{C}_i$ and $|\cdot|$ denotes the cardinal number of the set. Let $\mathbf{C}_{\mathcal{F}} = \{\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty} : \mathcal{C}_i \text{ be a flat CPC for all } i \in \mathbb{N} \cup \{0\}\}$. It is worth pointing out that a collection of flat CPCs $\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty}$ is the collection $\{T_{s_i}\}_{i=0}^{\infty}$ for some subsequence $\{s_i\}_{i=0}^{\infty}$ of $\mathbb{N} \cup \{0\}$. In [6], the authors prove that there exists a matrix A such that

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h_{top}^{\mathcal{C}}(\mathcal{T}_A) \in (\log 4, \log 5).$$

The result is different from the previous result of (4). However, we have the following result for tree-shifts defined on an unexpandable tree.

[2] The value $h_{top}^{\infty}(f)$ is also known as the *maximal pattern entropy* of f, introduced by Huang and Ye [26]. We refer the reader to [26] or [27] for more details and a complete bibliography.

[3] The sequence $\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty}$ symbolizes the sequence $S = \{s_n\}_{n=1}^{\infty}$ in $\mathbb{N}$ shifts.

Theorem 1.6. *Under the same assumptions as Theorem 1.2. Let*

$$\mathcal{H}^\infty(T) = \left\{ \sup_{\mathcal{C} \in \mathbf{C}_\mathcal{F}} h_{top}^{\mathcal{C}}(\mathcal{T}_A) : A \text{ is an irreducible matrix} \right\}.$$

Then $\mathcal{H}^\infty(T) = \{0, \log 2, \log 3, \ldots\}$.

The rest of this article is organized as follows. The proofs of Theorems 1.1 and 1.2 are provided in Section 2.1, the proofs of Theorems 1.3, 1.4 and 1.5 are presented in Section 2.2. The proof of Theorem 1.6 is outlined in Section 2.3. Finally, we conclude in Section 3.

2. Proofs of theorems

In this section, we present the proofs of theorems that are stated in Section 1. We assume that each ray in T has infinite cardinality.

2.1. Proofs of Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By Theorem 1.2, we have $h(\mathcal{T}_X) \leq h(X)$. Now, we aim to show that $h(\mathcal{T}_X) \geq h(X)$. Since $\lim_{n\to\infty} |T_n| < \infty$, we have $\lim_{n\to\infty} |T_n| = c$ for some constant c. As $|T_n|$ are positive integers, we have $c \in \mathbb{N}$. The existence of the limit implies that $N \in \mathbb{N}$ exists such that $|T_n| = c$ for all $n \geq N$. Fix a large number $n \geq N$. By the pigeonhole principle, there is a pattern $w \in \mathcal{P}([1, N+1], X)$ of length $N+1$ with at least $\frac{|\mathcal{P}([1,n+1],X)|}{|\mathcal{P}([1,N+1],X)|}$ continuations to legal patterns of length $n+1$. Now, decorating the first $N+1$ levels (Δ_N) of the tree with the pattern w, there are at least $\left(\frac{|\mathcal{P}([1,n+1],X)|}{|\mathcal{P}([1,N+1],X)|}\right)^c$ different ways to continue this to a decoration of Δ_n. That is,

$$|\mathcal{P}(\Delta_n, \mathcal{T}_X)| \geq \left(\frac{|\mathcal{P}([1, n+1], X)|}{|\mathcal{P}([1, N+1], X)|} \right)^c.$$

Then, we have

$$\begin{aligned} \frac{\log |\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|} &\geq \frac{\log \left(\frac{|\mathcal{P}([1,n+1],X)|}{|\mathcal{P}([1,N+1],X)|}\right)^c}{c(n+1)} \\ &= \frac{\log |\mathcal{P}([1, n+1], X)|}{n+1} - \frac{\log |\mathcal{P}([1, N+1], X)|}{n+1}. \end{aligned}$$

Taking $n \to \infty$, we have $h(\mathcal{T}_X) \geq h(X)$. The proof is complete. □

Proof of Theorem 1.2.

1. We first prove that if T is an unexpandable tree, then $h(\mathcal{T}_X) \leq h(X)$. Let $a_i = |T_i| - |T_{i-1}|$ for all $i \geq 1$ and $a_0 := |T_0| = 1$. For $n \geq 1$, the set Δ_n can be decomposed in the following manner. First, choose an arbitrary path from the root to a vertex in T_n, represented by the interval $[1, n+1]$. Similarly, choose a_1 paths of type $[1, n]$ from T_1 to T_n, avoiding the path chosen in the previous step. Proceeding inductively, we obtain a partition of Δ_n into a_{n-i+1} paths of type $[1, i]$ with $1 \leq i \leq n+1$. In this notation,

$$\Delta_n = \bigsqcup_{i=1}^{n+1} \bigsqcup_{j=1}^{a_{n-i+1}} [1, i].$$

We have

$$|\mathcal{P}(\Delta_n, \mathcal{T}_X)| \leq \prod_{i=1}^{n+1} |\mathcal{P}([1,i], X)|^{a_{n-i+1}}.$$

Then,

$$\begin{aligned} \frac{\log|\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|} &\leq \frac{\sum_{i=1}^{n+1} a_{n-i+1}\log|\mathcal{P}([1,i], X)|}{|\Delta_n|} \\ &= \frac{\sum_{i=1}^{n+1} a_{n-i+1}\log|\mathcal{P}([1,i], X)|}{\sum_{i=1}^{n+1} a_{n-i+1} i}. \end{aligned} \tag{6}$$

Fix an $\epsilon > 0$, since $\lim_{i\to\infty} \frac{\log|\mathcal{P}([1,i],X)|}{i} = \inf_{i\geq 1} \frac{\log|\mathcal{P}([1,i],X)|}{i} = h(X)$, there exists an $N > 0$ such that for $i \geq N$,

$$\log|\mathcal{P}([1,i], X)| \leq i(h(X) + \epsilon). \tag{7}$$

Then, by (6) and (7), we have

$$\begin{aligned} &\frac{\log|\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|} \\ &\leq \frac{\sum_{i=1}^{n+1} a_{n-i+1}\log|\mathcal{P}([1,i], X)|}{\sum_{i=1}^{n+1} a_{n-i+1} i} \\ &= \frac{\sum_{i=1}^{N} a_{n-i+1}\log|\mathcal{P}([1,i], X)|}{\sum_{i=1}^{n+1} a_{n-i+1} i} + \frac{\sum_{i=N+1}^{n+1} a_{n-i+1}\log|\mathcal{P}([1,i], X)|}{\sum_{i=1}^{n+1} a_{n-i+1} i} \\ &\leq \frac{\sum_{i=1}^{N} a_{n-i+1}\log|\mathcal{A}|^N}{\sum_{i=1}^{n+1} a_{n-i+1}} + \frac{\sum_{i=N+1}^{n+1} a_{n-i+1} i(h(X)+\epsilon)}{\sum_{i=N+1}^{n+1} a_{n-i+1} i} \\ &= N\log|\mathcal{A}| \frac{\sum_{i=n-N+1}^{n} a_i}{\sum_{i=0}^{n} a_i} + h(X) + \epsilon. \end{aligned} \tag{8}$$

Since T is an unexpandable tree, i.e., $\gamma_T = 1$, we have that for such $N > 0$,

$$\lim_{n\to\infty} \frac{|T_{n-N+1}|}{|T_n|} = 1.$$

Thus,

$$\lim_{n\to\infty} \frac{\sum_{i=n-N+1}^{n} a_i}{\sum_{i=1}^{n} a_i} = 0. \tag{9}$$

Combining (8) and (9), we have

$$\limsup_{n\to\infty} \frac{\log|\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|} \leq h(X) + \epsilon.$$

Since ϵ is arbitrary, we have

$$\limsup_{n\to\infty} \frac{\log|\mathcal{P}(\Delta_n, \mathcal{T}_X)|}{|\Delta_n|} \leq h(X).$$

2. We prove that if X is a subshift satisfying the almost specification property, then $h(\mathcal{T}_X) \geq h(X)$. Let c be the almost specification number of X. That is,

$$\forall u, v \in \mathcal{L}(X), \exists w \in \mathcal{L}(X) \text{ with } |w| \leq c \text{ such that } uwv \in \mathcal{L}(X).$$

Recall that for $n \geq 1$, the set Δ_n can be decomposed into a_{n-i+1} paths of type $[1,i]$ with $1 \leq i \leq n+1$. Given a path of type $[1,i]$ with $i \geq c$ and a word $v \in \mathcal{L}(X)$ of length $i-c$, we can choose (due to almost specification) a word w with $|w| = d \leq c$ such that the decoration of $[1,i]$ can start with wv. By the pigeon hole principle: Out of the $\mathcal{P}([1,i-c],X)$ legal words of length $i-c$, at least $\frac{|\mathcal{P}([1,i-c],X)|}{c}$ can start at the same position (d) and thus induce different decorations on $[1,i]$. Thus, for $n \geq c$, the number of patterns on Δ_n is greater than or equal to the product of the numbers of patterns on $[1,i]$ ($c+1 \leq i \leq n+1$). That is,

$$\prod_{i=c+1}^{n+1} \left(\frac{|\mathcal{P}\left([1,i-c],X\right)|}{c} \right)^{a_{n-i+1}} \leq |\mathcal{P}\left(\Delta_n, \mathcal{T}_X\right)|. \tag{10}$$

Then,

$$\begin{aligned}
\frac{\log|\mathcal{P}\left(\Delta_n,\mathcal{T}_X\right)|}{|\Delta_n|} &\geq \frac{\sum_{i=c+1}^{n+1} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{|\Delta_n|} - \frac{|T_n|\log c}{|\Delta_n|} \\
&= \frac{\sum_{i=c+1}^{n+1} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{\sum_{i=1}^{n+1} a_{n-i+1} i} - \frac{|T_n|\log c}{|\Delta_n|} \\
&= \frac{\sum_{i=c+1}^{n+1} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{\sum_{i=1}^{c} a_{n-i+1} i + \sum_{i=c+1}^{n} a_{n-i+1} i} - \frac{|T_n|\log c}{|\Delta_n|} \\
&\geq \frac{\sum_{i=c+1}^{n+1} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{c\sum_{i=1}^{n+1} a_{n-i+1} + \sum_{i=c+1}^{n+1} a_{n-i+1}(i-c)} - \frac{|T_n|\log c}{|\Delta_n|}.
\end{aligned}$$

Since $\gamma_T = 1$, we have

$$\lim_{n\to\infty} \frac{c\sum_{i=1}^{n+1} a_{n-i+1}}{\sum_{i=c+1}^{n+1} a_{n-i+1}(i-c)} = \lim_{n\to\infty} \frac{cT_n}{T_{n-c} + T_{n-1-c} + \cdots + T_1 + T_0} = 0.$$

Hence, taking $n \to \infty$, we obtain that

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log|\mathcal{P}\left(\Delta_n,\mathcal{T}_X\right)|}{|\Delta_n|} &\geq \liminf_{n\to\infty} \frac{\sum_{i=c+1}^{n+1} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{c\sum_{i=1}^{n+1} a_{n-i+1} + \sum_{i=c+1}^{n+1} a_{n-i+1}(i-c)} \\
&= \liminf_{n\to\infty} \frac{\sum_{i=c+1}^{n} a_{n-i+1}\log|\mathcal{P}\left([1,i-c],X\right)|}{\sum_{i=c+1}^{n} a_{n-i+1}(i-c)} \\
&\geq \inf_{i\geq 1} \frac{\log|\mathcal{P}\left([1,i],X\right)|}{i} \\
&= h(X).
\end{aligned}$$

Then, combining **1.**, we have $h(\mathcal{T}_X) = h(X)$. The proof is complete. □

We recall that a shift space is termed a *coded system* if it can be presented by an irreducible directed graph (with a countable number of vertices) whose edges are labeled by symbols from a finite alphabet $\mathcal{A}$. In this context, "presented" signifies that the shift space is the closure in $\mathcal{A}^{\mathbb{N}\cup\{0\}}$, serving as labels for infinite paths in the graph. It is well known that X being a coded system equivalent to X being the closure of the set of sequences obtained by freely concatenating the words (or generators) in a (possibly infinite) list of words over a finite alphabet ([33, Section 13.5]). Given that irreducible SFTs and coded systems with finite generators satisfy the almost specification property, we obtain the following result.

Corollary 2.1. *If $X \subseteq \mathcal{A}^{\mathbb{N}}$ is an SFT or a coded system with finite generators, then $h(\mathcal{T}_X) = h(X)$.*

Proof.

1. Let X be an SFT with the adjacency matrix A. Suppose $A_1, \ldots, A_k$ are the irreducible components of A, and $1 \leq q \leq k$ is the integer such that $\rho(A_q) = \max_{1\leq i\leq k} \rho(A_i)$, where $\rho(A)$ denotes the maximum eigenvalue of the matrix A. We aim to prove that $h(\mathcal{T}_A) = \log \rho(A_q)$. Note that $h(\mathcal{T}_{A_q}) \leq h(\mathcal{T}_A)$ because $\mathcal{T}_{A_q} \subseteq \mathcal{T}_A$. Since A_q is irreducible, the subshift X with the adjacency matrix A_q satisfies the almost specification property. By Theorem 1.2, we have $h(\mathcal{T}_{A_q}) = h(A_q)$ and thus $h(\mathcal{T}_A) \geq h(A_q)$. On the other hand, we have $h(\mathcal{T}_A) \leq h(A)$ by Theorem 1.2. Since $h(A) = \max_{1\leq i\leq k} \rho(A_i) = \rho(A_q) = h(A_q)$, we then have $h(\mathcal{T}_A) \leq h(A_q)$. The proof is complete.

2. If X is a coded system with finite generators, then the proof is easily attained by Theorem 1.2 because X satisfies the almost specification property. □

2.2. **Proofs of the Theorems 1.3, 1.4 and 1.5.**

Proof of Theorem 1.3.

1. Since for any $a \in \mathcal{A}$,

$$\mathcal{P}_a(\Delta_n, \mathcal{T}_X) \subseteq \mathcal{P}(\Delta_n, \mathcal{T}_X),$$

we have

$$|\mathcal{P}_a(\Delta_n, \mathcal{T}_X)| \leq |\mathcal{P}(\Delta_n, \mathcal{T}_X)|.$$

Thus, $h_a(\mathcal{T}_X) \leq h(\mathcal{T}_X) = h(X)$ by Corollary 2.1. It remains to show that $h_a(\mathcal{T}_X) \geq h(X)$. Since X is an irreducible SFT, by the similar argument of Theorem 1.2, we have $h_a(\mathcal{T}_X) \geq h(X)$. Thus, $h_a(\mathcal{T}_X) = h(X) = h(\mathcal{T}_X)$. The limit of the definition defining $h_a(\mathcal{T}_X)$ exists.

2. The proof is obtained directly from Corollary 2.1. □

It is worth mentioning that the proof of Theorem 1.3 (1) also works if the shift X satisfies almost specification property.

In order to prove Theorem 1.4, we need the following theorem.

Theorem 2.2 ([18, Theorem 3.3])**.** *Let X be an SFT with $h(X) > 0$. Then there exists a family of SFT subsystems of X whose entropies are dense in* $[0, h(X)]$.

Proof of Theorem 1.4.

1. Since $h(\mathcal{T}_X) > 0$ and X is an SFT, we have $h(X) = h(\mathcal{T}_X) > 0$ by Corollary 2.1. Since $h(X) > 0$ and X is an SFT, there exists a set $\{X_i : i \in \mathbb{N}\}$ of subSFT of X such that $\{h(X_i) : i \in \mathbb{N}\}$ is dense in $[0, h(X)]$ by Theorem 2.2. Since X_i is an SFT, then by Corollary 2.1 again, we have $h(\mathcal{T}_{X_i}) = h(X_i)$. This implies $\{h(\mathcal{T}_{X_i}) : i \in \mathbb{N}\}$ is dense in $[0, h(\mathcal{T}_X)]$. Note that $\mathcal{T}_{X_i} \subseteq \mathcal{T}_X$ is a tree-subSFT for all $i \in \mathbb{N}$. The proof is complete.

2. By Corollary 2.1, we have $h(A) = h(\mathcal{T}_A)$. Then, the proof follows immediately from the corresponding (well-known) result for 1-dimensional SFTs. □

Proof of Theorem 1.5. The proof is clearly obtained by Corollary 2.1 and [32, Theorem 3], which are stated in Section 1. □

2.3. **Proof of the Theorem 1.6.**

In order to prove Theorem 1.6, the following definitions are needed. Let $\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty}$ be a sequence of finite subsets of T. We say that $\mathcal{C}_i$ is a *complex prefix code* (CPC) on T if, for every ray $\mathcal{R} \subseteq T$, the cardinality $|\mathcal{R} \cap \mathcal{C}_i|$ of $\mathcal{R} \cap \mathcal{C}_i$ is equal to 1. We say a CPC $\mathcal{C}_i$ is *flat* if every member belongs to $\mathcal{C}$ having the same length. For a sequence of flat CPCs $\mathcal{C} = \{\mathcal{C}_i\}_{i=0}^{\infty}$, the *topological* *sequence entropy* $h^{\mathcal{C}}_{top}(\mathcal{T}_X)$ is defined in (5). We also define the *topological surface entropy* by

$$h^{\mathcal{L}}_{top}(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log |\mathcal{P}(T_n, \mathcal{T}_X)|}{|T_n|}.$$

The following lemma is a general version of [6, Propositions 3.2 and Corollary 3.3]. The proof of the lemma is already shown in [6], but for the self-containment of this article, we provide the proof here.

Lemma 2.3. *Let $\mathcal{T}_X$ be a hom tree-shift with $\lim_{n\to\infty} |T_n| = \infty$. Then,*

$$\sup_{\mathcal{C}\in\mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_X) = h^{\mathcal{L}}_{top}(\mathcal{T}_X).$$

Proof. We first prove that $h^{\mathcal{C}}(\mathcal{T}_X) \le h^{\mathcal{L}}_{top}(\mathcal{T}_X)$ for all $\mathcal{C} \in \mathbf{C}_{\mathcal{F}}$. Since

$$\left|\mathcal{P}\left(\Delta^{\mathcal{C}}_n, \mathcal{T}_X\right)\right| \le \prod_{i=0}^{n-1} |\mathcal{P}(\mathcal{C}_i, \mathcal{T}_X)| \quad \text{and} \quad \left|\Delta^{\mathcal{C}}_n\right| = \left|\cup_{i=0}^{n-1}\mathcal{C}_i\right| = \sum_{i=0}^{n-1} |\mathcal{C}_i|,$$

then for all $n \ge 0$,

$$\frac{\log\left|\mathcal{P}\left(\Delta^{\mathcal{C}}_n, \mathcal{T}_X\right)\right|}{|\Delta^{\mathcal{C}}_n|} \le \frac{\sum_{i=0}^{n-1} \log |\mathcal{P}(T_{s_i}, \mathcal{T}_X)|}{\sum_{i=0}^{n-1} |T_{s_i}|},$$

where $s_i \ge 0$ satisfies $\mathcal{C}_i = T_{s_i}$ for all $i \ge 0$.

By taking the limit from both sides of the inequality, we have $h^{\mathcal{C}}(\mathcal{T}_X) \le h^{\mathcal{L}}_{top}(\mathcal{T}_X)$. Since $\mathcal{C} \in \mathbf{C}_{\mathcal{F}}$ is arbitrary, we have

$$\sup_{\mathcal{C}\in\mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_X) \le h^{\mathcal{L}}_{top}(\mathcal{T}_X).$$

Conversely, since $h^{\mathcal{L}}_{top}(\mathcal{T}_X) = \limsup_{n\to\infty} \frac{\log|\mathcal{P}(T_n,\mathcal{T}_X)|}{|T_n|}$, there exists a sequence $\{s_i\}_{i=0}^{\infty}$ such that $h^{\mathcal{L}}_{top}(\mathcal{T}_X) = \lim_{i\to\infty} \frac{\log|\mathcal{P}(T_{s_i},\mathcal{T}_X)|}{|T_{s_i}|}$. Since $\lim_{n\to\infty} |T_n| = \infty$, there exists a subsequence $\{s'_i\}_{i=0}^{\infty}$ of $\{s_i\}_{i=0}^{\infty}$ such that $\lim_{n\to\infty} \frac{\left|T_{s'_{n-1}}\right|}{\sum_{i=0}^{n-1}\left|T_{s'_i}\right|} = 1$.

Since for $n \ge 1$, $\left|\mathcal{P}\left(T_{s'_{n-1}}, \mathcal{T}_X\right)\right| \le \left|\mathcal{P}\left(\Delta^{\mathcal{C}}_n, \mathcal{T}_X\right)\right|$ where $\mathcal{C}_i = T_{s'_i}$ for all $i \ge 0$, we have

$$\begin{aligned}
h^{\mathcal{C}}_{top}(\mathcal{T}_X) &= \limsup_{n\to\infty} \frac{\log\left|\mathcal{P}\left(\Delta^{\mathcal{C}}_n, \mathcal{T}_X\right)\right|}{|\Delta^{\mathcal{C}}_n|} \\
&\ge \limsup_{n\to\infty} \frac{\log\left|\mathcal{P}\left(T_{s'_{n-1}}, \mathcal{T}_X\right)\right|}{|\Delta^{\mathcal{C}}_n|} \\
&= \limsup_{n\to\infty} \frac{\log\left|\mathcal{P}\left(T_{s'_{n-1}}, \mathcal{T}_X\right)\right|}{\left|T_{s'_{n-1}}\right|} \frac{\left|T_{s'_{n-1}}\right|}{|\Delta^{\mathcal{C}}_n|} \\
&= h^{\mathcal{L}}_{top}(\mathcal{T}_X).
\end{aligned}$$

Thus, $\sup_{\mathcal{C}\in\mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}_{top}(\mathcal{T}_X) \ge h^{\mathcal{L}}_{top}(\mathcal{T}_X)$. The proof is complete. □

Let A be a nonnegative matrix. The *period of a state* i, denoted by $\mathrm{per}(i)$, is the greastest common divisor of those integers $n \ge 1$ for which $(A^n)_{i,i} > 0$. The *period* $\mathrm{per}(A)$ of the matrix A is the greatest common divisor of the numbers $\mathrm{per}(i)$ that are finite or is ∞ if $\mathrm{per}(i) = \infty$ for all i. The following result is needed.

Proposition 2.4 ([33, Proposition 4.5.6])**.** *Let $A \neq [0]$ be irreducible with period p. Then there are exactly p period classes, which can be ordered as $D_0, D_1, \ldots, D_{p-1}$ so that every edge that starts in D_i terminates in D_{i+1} (or in D_0 if $i = p-1$).*

Theorem 1.6 is a corollary of the following theorem. Before stating the theorem, we need more definitions. For $m \geq 1$, the *m-block representation $A^{[m]}$ of a matrix* A is defined by deleting zero rows and columns of the following matrix

$$B = [B_{i,j}] \text{ with } B_{i,j} = \begin{cases} 1 & \text{, if } \prod_{\ell=1}^{m} A_{x_\ell, x_{\ell+1}} = 1, \\ 0 & \text{, if } \prod_{\ell=1}^{m} A_{x_\ell, x_{\ell+1}} = 0. \end{cases}$$

for all $i = x_1 \cdots x_m$ and $j = x_2 \cdots x_{m+1}$, where x_i is state of A. The collection of all states of $A^{[m]}$ is called *the m-block representation of state set of A.*

Theorem 2.5. *If $\gamma_T = 1$ and A is an irreducible 0-1 matrix with period p, then we have the following assertions.*

1.

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_A) = \max_{0 \leq i \leq p-1} \log |D_i|,$$

where D_i is a periodic class of A.

2. If $m \geq 1$, then

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_{A^{[m]}}) = \max_{0 \leq i \leq p-1} \log \left| D_i^{[m]} \right|,$$

where $A^{[m]}$ is an m-block representation of A and $D_i^{[m]}$ is an m-block representation of periodic class D_i.

3. If Y is a zero block ϕ factor of A (i.e., $\phi : X \to Y$ is a zero sliding block code), then

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_Y) = \max_{0 \leq i \leq p-1} \log |\phi(D_i)|,$$

where D_i is a periodic class of A.

Proof.

1. Since A is an irreducible matrix with periodic p, we may assume that A is of the following form (after permutation of the basis vectors). Then Proposition 2.4 implies that

$$A = \begin{bmatrix} 0 & B_0 & 0 & \cdots & 0 \\ 0 & 0 & B_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_{p-2} \\ B_{p-1} & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$A^p = \begin{bmatrix} A_0 & 0 & 0 & \cdots & 0 \\ 0 & A_1 & 0 & \cdots & 0 \\ 0 & 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{p-1} \end{bmatrix},$$

where $A_i = B_i \cdots B_{p-1} B_0 \cdots B_{i-1}$ and A_i is a primitive matrix for all $0 \leq i \leq p-1$.

Let N_i be the primitive number of A_i for all $0 \leq i \leq p-1$, and let $N = \max_{0 \leq i \leq p-1} N_i$. Denote by D_i the periodic class of A concerning B_i. Then, B_i is a $|D_i|$ by $|D_{i+1}|$ matrix for all $0 \leq i \leq p-2$ and B_{p-1} is a $|D_{p-1}|$ by $|D_0|$ matrix. Thus, A_i is a $|D_i|$ by $|D_i|$ matrix for all $0 \leq i \leq p-1$. Since A is of the above form, we have that all symbols at the same level of a pattern in $\mathcal{T}_A$ are all in the same periodic class D_i for some $0 \leq i \leq p-1$. Thus, we easily have the upper bound of $|\mathcal{P}(\{T_{s_i}\}_{i=0}^n, \mathcal{T}_A)|$, that is, for all $n \geq 0$,

$$|\mathcal{P}(\{T_{s_i}\}_{i=0}^n, \mathcal{T}_A)| \leq p \left(\max_{0 \leq i \leq p-1} |D_i| \right)^{\sum_{i=0}^n |T_{s_i}|}, \forall \{s_i\}_{i=0}^\infty \subseteq \mathbb{N} \cup \{0\}. \tag{11}$$

For the lower bound, if $\lim_{n\to\infty} |T_n| < \infty$, similar to the proof of Theorem 1.1, there exist $N_1, c \in \mathbb{N}$ such that for $n \geq N_1$, $|T_n| = c$. We choose a subsequence $\{s_i\}_{i=0}^\infty$ of $\mathbb{N} \cup \{0\}$ defined by $s_i = pN_1N(i+1)$, then we have T_{s_i} and T_{s_j} are in the same periodic class for all $i, j \in \mathbb{N} \cup \{0\}$, and $s_{i+1} - s_i \geq pN$ for all i, and $s_0 \geq N_1$. Then, we decorate the level T_{s_0} by a member in D_k with $|D_k| = \max_{0 \leq j \leq p-1} |D_j|$, and then, by the primitivity of A^p, we obtain that c members in T_{s_i} can be arbitrarily chosen (represented as decorated symbols) from D_k. Thus,

$$|\mathcal{P}(\{T_{s_i}\}_{i=0}^n, \mathcal{T}_A)| \geq \left(\max_{0 \leq i \leq p-1} |D_i| \right)^{\sum_{i=1}^n c}. \tag{12}$$

Combining (11) and (12), we have

$$\begin{aligned} \frac{\log p \left(\max_{0 \leq i \leq p-1} |D_i|\right)^{\sum_{i=0}^{n-1} |T_{s_i}|}}{\sum_{i=0}^{n-1} |T_{s_i}|} &\geq \frac{\log \left|\mathcal{P}\left(\{T_{s_i}\}_{i=0}^{n-1}, \mathcal{T}_A\right)\right|}{\sum_{i=0}^{n-1} |T_{s_i}|} \\ &\geq \frac{\log \left(\max_{0 \leq i \leq p-1} |D_i|\right)^{c(n-1)}}{cn}. \end{aligned}$$

Taking $n \to \infty$, we have

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_A) = \max_{0 \leq i \leq p-1} \log |D_i|.$$

Now, we prove the case $\lim_{n\to\infty} |T_n| = \infty$. By similar reasoning as in (11), we have

$$|\mathcal{P}(T_n, \mathcal{T}_A)| \leq p \left(\max_{0 \leq i \leq p-1} |D_i| \right)^{|T_n|}, \forall n \in \mathbb{N} \cup \{0\}. \tag{13}$$

For the lower bound, we decorate the level T_{n-pN} with a member from D_i where $|D_i| = \max_{0 \leq j \leq p-1} |D_j|$. By the primitivity of A^p, we can arbitrarily choose (decorate symbol) at least $|T_{n-pN}|$ members in T_n from D_i. This implies

$$|\mathcal{P}(T_n, \mathcal{T}_A)| \geq \left(\max_{0 \leq i \leq p-1} |D_i| \right)^{|T_{n-pN}|}. \tag{14}$$

Combining (13) and (14), we have

$$\frac{\log p \left(\max_{0 \leq i \leq p-1} |D_i|\right)^{|T_n|}}{|T_n|} \geq \frac{\log |\mathcal{P}(T_n, \mathcal{T}_A)|}{|T_n|} \geq \frac{\log \left(\max_{0 \leq i \leq p-1} |D_i|\right)^{|T_{n-pN}|}}{|T_n|}.$$

Since $\gamma_T = 1$, we have $\lim_{n\to\infty} \frac{|T_{n-pN}|}{|T_n|} = 1$. This implies that $h_{\text{top}}^{\mathcal{L}}(\mathcal{T}_A)$ exists and

$$h_{top}^{\mathcal{L}}(\mathcal{T}_A) = \max_{0 \leq i \leq p-1} \log |D_i|. \tag{15}$$

Since $\lim_{n\to\infty} |T_n| = \infty$, by Lemma 2.3 and (15), we have

$$\sup_{\mathcal{C} \in \mathbf{C}_{\mathcal{F}}} h^{\mathcal{C}}(\mathcal{T}_A) = \max_{0 \leq i \leq p-1} \log |D_i|.$$

2. Since A is irreducible with period p, then $A^{[m]}$ is irreducible with period p. We can apply spectral decomposition and a similar estimate as in the proof of **1**.

3. Since the zero sliding block code ϕ maps symbols to symbols, the proof follows a similar argument as given in **1**. □

Proof of Theorem 1.6. The proof is obtained directly by Theorem 2.5 (1) with the cardinality $|D_i|$ of the periodic class D_i of an irreducible 0-1 matrix A, which is a positive integer for all $0 \leq i \leq p-1$. □

3. Conclusion

In this section, we summarize the results of this article. Concerning topological entropy, we establish that the entropy $h(\mathcal{T}_X)$ on an unexpandable tree is equal to the entropy $h(X)$ of a base shift X when X is a subshift satisfying the almost specification property (Theorem 1.2). Additionally, we derive some fundamental properties, including entropy approximation (Theorem 1.4) and the denseness of entropy of the subsystems (Theorem 1.5). Regarding topological sequence entropy, we show that the set of sequence entropies of hom tree-shifts with base shift, generated by an irreducible matrix A, is $\{0, \log 2, \log 3, ...\}$ (Theorem 1.6). More precisely, these entropies correspond to the logarithms of the largest cardinalities of the periodic classes of A (Theorem 2.5).

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