EXERCISE 1

1. Demonstrate that the large deviation principle may not hold for all measurable sets without taking closure and interior.
2. Show Remark 1.4.
3. Show Remark 1.6.
4. Prove the Cramér's theorem for i.i.d. random variables with finite alphabet (Theorem 2.7).
5. Prove Gibb's inequality: Given any finite alphabet Σ and $\mu, \nu \in M_1(\Sigma)$, we have $0 \leq H(\nu|\mu)$ with equality holding if and only if $\mu = \nu$. (Hint: using convexity of $x \log x$.)

1. Let $\mu_n = \delta_{\frac{1}{n}}$. Suppose $I : \mathbb{R} \to \mathbb{R}$ is a rate function associated with the modified LDP:

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(A) \le \inf_{x\in A} I(x),$$
$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(A) \ge \inf_{x\in \mathring{A}} I(x).$$

By plugging $A = [a,b] \not\supseteq \{0\}$ into the first and the second inequality, respectively, we obtain that $I(0) \ge 0$ and that $I(x) = -\infty$ for all $x \ne 0$. However, if plugging $A = (0,1)$ into the first inequality, we have 0 on the left-hand side and $-\infty$ on the right-hand side, a contradiction. A similar argument demonstrate that the following may fail for an obvious reason:

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(A) \le \inf_{x\in \overline{A}} I(x),$$
$$\liminf_{n\to\infty} \frac{1}{n} \log \mu_n(A) \ge \inf_{x\in A} I(x).$$

2. Suppose $\mathcal{B}_{\mathcal{X}} \subseteq \mathcal{B}$. If it holds for all closed sets and open sets, then obviously it holds for all measurable set. If it holds for all measurable sets, then it holds for all measurable sets.
3. Suppose μ_ε is exponentially tight. If F is a closed set, then

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F) \le -\inf_{x\in F} I(x)$$

For all $\infty > a > 0$, there exists a compact set K_a such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(K_a^c) < -a.$$

Hence,

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F) &\le \max\left\{\limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F\cap K_a), \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(K_a^c)\right\} \\ &= \lim_{a\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(F\cap K_a) \le -\inf_{x\in F\cap K_a} I(x),\end{aligned}$$

where the last inequality holds as $F \subset K_a$ is compact due to the Hausdorff assumption.
4. If $I : \mathbb{R}^d \to \mathbb{R}$ is a rate function associated with the LDP for measures μ_n on $\mathbb{R}^d$, then given any continuous function $f : \mathbb{R}^d \to \mathbb{R}$, the measures $\mu_n \circ f^{-f}$ satisfies LDP with rate

$$I_f(x) := \inf\{I(y) : f(y) = x\}.$$

In this case,

$$\begin{aligned}I_f(x) &= \inf_{\nu:\langle \mathbf{f},\nu\rangle = x} \sum_a \nu(a) \log \frac{\nu(a)}{\mu(a)} = \inf_{\nu} \sup_{\lambda\in\mathbb{R}} \sum_a \nu(a) \log \frac{\nu(a)}{\mu(a)} - \lambda(\langle \mathbf{f},\nu\rangle - x) \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{\nu} \lambda x + \sum_a \nu(a) \log \frac{\nu(a)}{\mu(a) e^{\lambda f(a)}} = \sup_{\lambda\in\mathbb{R}} \inf_{\nu} \lambda x + \sum_a \nu(a) \log \frac{\nu(a)}{\mu(a) e^{\lambda f(a)}} \\ &= \sup_{\lambda\in\mathbb{R}} \inf_{\nu} \lambda x - \Lambda(\lambda),\end{aligned}$$

where

$$\Lambda(\lambda) = \log \sum_a \mu(a) e^{\lambda f(a)}.$$

Alternatively, observe that

$$\Lambda(\lambda) \geq \lambda \langle \mathbf{f}, \nu \rangle - H(\nu|\mu)$$

with equality holding when $\nu_\lambda(a) = \mu(a) e^{\lambda f(a) - \Lambda(\lambda)}$. Hence,

$$H(\nu|\mu) \geq \lambda x - \Lambda(\lambda).$$

Since Λ is convex, so is $\Lambda(\lambda) - \lambda x$ and

$$\Lambda'(\lambda) = (\mathbf{f}, \nu_\lambda) = x \Rightarrow \Lambda^*(x) = \lambda x - \Lambda(\lambda) = \sup_{\eta \in \mathbb{R}} \eta x - \Lambda(\eta),$$

we have

$$I(x) = \sup_{\lambda \in \mathbb{R}} \lambda x - \Lambda(\lambda) \text{ for } x \in \{\Lambda'(\lambda) : \lambda \in \mathbb{R}\}.$$

where the domain is $(\min_a f(a), \max_a f(a))$ since Λ' is strictly increasing. For the endpoint $\min_a f(a)$ (similarly for $\max_a f(a)$), consider ν with $\nu(a_{\min}) = 1$ so that

$$-\log \mu(a_{\min}) = H(\nu|\mu) \geq I(x) \geq \lim_{\lambda \to \infty} [\lambda x - \Lambda(\lambda)] = -\log \mu(a_{\min})$$

This finishes the proof.

5. Note that if $\sum_i a_i = \sum_i b_i = 1$

$$\sum_i a_i \log \frac{a_i}{b_i} = \left(\sum_j b_j \right) \sum_i \frac{b_i}{\sum_j b_j} \frac{a_i}{b_i} \log \frac{a_i}{b_i} \geq 0.$$