EXERCISE 3

1. Prove Remark 4.6.
2. Prove the converse of existence of LDP (Proposition 4.7).
3. Prove by an application of Fatou's lemma that any logarithmic moment generating function is lower semicontinuous.
4. Suppose that $\mu = f dx \in M_1\left(\mathbb{R}^d\right)$ admits a density $f > 0$. If f is continuous at x with $f(x) > 0$, then $\Lambda^*(x) < \infty$.
5. Suppose that $\mu = f dx \in M_1\left(\mathbb{R}^d\right)$ possesses the density
$$f(x) = \begin{cases} x & \text{for } x \in [0,1], \\ 2 - x & \text{for } x \in [1,2]. \end{cases}$$
Prove that $\Lambda^*(0) = \infty$ while $\mu((-\delta, \delta)) > 0$ for all $\delta > 0$.

1. We begin by noting the fact that I is indeed a rate function, which follows naturally from definition.

   For the lower bound in LDP, observe that for every open set G and any $x \in G$, there exists $A \in \mathcal{A}$ such that $x \in A \subset G$ such that
   $$\underline{\mathcal{L}}_G \geq \underline{\mathcal{L}}_A \geq -\underline{I}(x),$$
   from which the desired lower bound follows naturally.

   For upper bound, given any compact set F, we can find for every point x a neighborhood A_x satisfying $\overline{\mathcal{L}}_{A_x} \leq -\overline{I}^\delta(x)$. By compactness, it admits some finite open subcover $\{A_{x_i}\}_{i=1}^m$, and thus
   $$\overline{\mathcal{L}}_F \leq \max_{1\leq i\leq m} \overline{\mathcal{L}}_{A_i} \leq \max_{1\leq i\leq m} -\overline{I}^\delta(x_i) \leq -\inf_{x\in F} \overline{I}^\delta(x).$$
   This finishes the proof.
2. Suppose $\underline{I}(x) > \overline{I}(x)$ for some x. By lower semicontinuity of I and regularity of $\mathcal{X}$, for every $\delta > 0$ there exists an open neighborhood A of x such that
   $$-I^\delta(x) \geq -\inf_{y\in\overline{A}} I(y) \geq \overline{\mathcal{L}}_A \geq -\overline{I}(x).$$
   On the other hand, by definition, for every $\delta > 0$ there exists an open neighborhood A of x such that
   $$-I(x) \leq -\inf_{y\in\mathring{A}} I(y) \leq \underline{\mathcal{L}}_A \leq -\underline{I}^\delta(x).$$
   Combining the inequalities proves the proposition.
3. Clearly,
   $$\liminf_{\lambda\to\lambda_0} \log \mathbb{E}\left[e^{\langle\lambda,x\rangle}\right] \geq \log \mathbb{E}\left[e^{\langle\lambda_0,x\rangle}\right].$$
4. Obviously, there exists a δ-ball $B_\delta(x)$ at x on which $f|_{B_\delta(x)} \geq \frac{f(x)}{2}$. Hence, by writing $H_\lambda = \{y : \langle\lambda, y-x\rangle \geq 0\}$
   $$\begin{aligned}\Lambda^*(x) &\leq \sup_{\lambda\in\mathbb{R}^d}\left[-\log\int_{B_\delta(x)\cap H_\lambda} e^{\langle\lambda,y-x\rangle} f(y)dy\right] \\ &\leq \sup_{\lambda\in\mathbb{R}^d} -\log\left(\frac{|B_\delta(x)|}{2}\cdot\frac{f(x)}{2}\right) < \infty.\end{aligned}$$
5. Indeed,
   $$\begin{aligned}\Lambda^*(0) &\geq \sup_{\lambda\in\mathbb{R}}\left[-\log\int_0^1 e^{\lambda x} f(x)dx\right] \\ &\geq \lim_{\lambda\to-\infty}\left[-\log\int_0^1 2e^{\lambda x}dx\right] \\ &= \lim_{\lambda\to-\infty} -\log\left(\frac{2e^\lambda - 2}{\lambda}\right) = \infty.\end{aligned}$$