EXERCISE 4

1. For multivariate normal distribution $X_1 \sim \text{Normal}(\mathbf{0}, \mathbf{\Sigma})$, determine its LDP.
2. Consider a family of real valued random variables $\{Z_\varepsilon\}$, where $\mathbb{P}(Z_\varepsilon = 0) = 1 - 2p_\varepsilon$, $\mathbb{P}(Z_\varepsilon = m_\varepsilon) = \mathbb{P}(Z_\varepsilon = -m_\varepsilon) = p_\varepsilon$.
   - Prove that if $\lim_{\varepsilon \to 0} \varepsilon \log p_\varepsilon = -\infty$, then the laws of $\{Z_\varepsilon\}$ are exponentially tight and satisfies the LDP with the convex, good rate function
   $$I(x) = \begin{cases} 0 & x = 0, \\ \infty & \text{otherwise.} \end{cases}$$
   - Let $m_\varepsilon = -\varepsilon \log p_\varepsilon$ and define $\Lambda(\lambda) = \lim_{\varepsilon \to 0} \varepsilon \mathbb{E}\left[e^{\lambda Z_\varepsilon / \varepsilon}\right]$. Prove that
   $$\Lambda(\lambda) = \begin{cases} 0 & |\lambda| \leq 1, \\ \infty & \text{otherwise,} \end{cases}$$
   and its Fenchel-Legendre transform is $\Lambda^*(x) = |x|$.
   - Observe that $\Lambda(\lambda) \neq \sup_{x \in \mathbb{R}} \{\lambda x - I(x)\}$ and $\Lambda^* \neq I$.
3. Let $X_1, ..., X_n, ...$ be a real-valued, zero mean, stationary Gaussian process with covariance sequence $R_i = \mathbb{E}(X_n X_{n+i})$. Suppose the process has a finite power P defined via $P = \lim_{n \to \infty} \sum_{i=-n}^{n} R_i \left(1 - \frac{|i|}{n}\right)$. Let μ_n be the law of the empirical mean $\hat{S}_n$ of the first n samples of this process. Prove that $\{\mu_n\}$ satisfy the LDP with the good rate function $\Lambda^*(x) = \frac{x^2}{2P}$.

1. We first show the LDP for $X_1 \sim \text{Normal}(\mathbf{0}, I)$ admits a pdf
$$f(\mathbf{x}) = (2\pi)^{-\frac{k}{2}} \exp\left(-\frac{1}{2}\, \|x\|^2\right).$$
In this case, the logarithmic moment generating function is
$$\log \int e^{\langle \lambda, \mathbf{x}\rangle} f(\mathbf{x}) d\mathbf{x} = \log\left[\prod_{i=1}^{d} e^{\frac{\lambda_i^2}{2}} \int e^{-\frac{(x-\lambda_i)^2}{2}} dx\right] = \frac{\|\lambda\|^2}{2}.$$
By solving the quadratic optimization problem, we have
$$\sup_{\lambda \in \mathbb{R}^d} \left[\langle \lambda, \mathbf{x}\rangle - \frac{\|\lambda\|^2}{2}\right] = \frac{\|\mathbf{x}\|^2}{2}.$$
Now if $\mathbf{\Sigma} = A^T D A$ for some orthogonal matrix A and (positively) diagonal matrix D, we have $AD^{\frac{1}{2}} X_1 \sim \text{Normal}(\mathbf{0}, \mathbf{\Sigma})$. By the contraction principle, it gives the rate function $I(\mathbf{x}) = \frac{\mathbf{x}^T \Sigma^{-1} \mathbf{x}}{2}$.
2. Regardless of m_ε, for every $M > 0$ we have, under the assumption,
$$\limsup_{\varepsilon \to 0} \varepsilon \mathbb{P}(|Z_\varepsilon| > M) \leq \limsup_{\varepsilon \to 0} \varepsilon \log(2p_\varepsilon) = -\infty,$$

proving that Z_ε is exponentially tight. Now given any set A, we have that
$$\limsup_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(A) = \begin{cases} 0 & \text{if } 0 \in A, \\ -\infty & \text{otherwise.} \end{cases}$$
Since I is clearly a good rate function, the property above shows that Z_ε satisfies the LDP with the rate I. Now that
$$\Lambda_\varepsilon(\lambda) = \log \mathbb{E}\left[e^{\langle \lambda, Z_\varepsilon\rangle}\right] = \log\left(1 - 2p_\varepsilon + p_\varepsilon e^{\lambda m_\varepsilon} + p_\varepsilon e^{-\lambda m_\varepsilon}\right)$$
If $\limsup p_\varepsilon$
$$\Lambda(\lambda) = \limsup_{\varepsilon \to 0} \varepsilon \Lambda_\varepsilon\left(\frac{\lambda}{\varepsilon}\right) = \limsup_{\varepsilon \to 0} \varepsilon \log\left(1 - 2p_\varepsilon + p_\varepsilon e^{\frac{\lambda m_\varepsilon}{\varepsilon}} + p_\varepsilon e^{-\frac{\lambda m_\varepsilon}{\varepsilon}}\right)$$
$$= \max\left(0, \limsup_{\varepsilon \to 0} \varepsilon(1 - |\lambda|) \log p_\varepsilon\right).$$
The proposed expression holds under the assumption $\varepsilon \log p_\varepsilon \to -\infty$. By taking the Legendre transform, we have that
$$\Lambda^*(x) = \sup_{\lambda \in \mathbb{R}} \lambda x - \Lambda(\lambda) = |x|.$$
Finally, observe that $I^*(\lambda) = 0$ for all λ.
3. To this end, observe that the vector $(X_1, X_2, ..., X_n)$ follows a multivariate Gaussian distribution with covariance matrix $\mathbf{\Sigma}_n = \left(R_{i-j}\right)_{ij}$, where $R_{-i} = R_i$ for all $i \in \mathbb{N}$. To simplify the discussion, we make use of the fact that $n\hat{S}_n$ is a normal distribution with variance
$$\mathbb{E}\left[\left(n\hat{S}_n\right)^2\right] = n \sum_{i=-n}^{n} \left(1 - \frac{|i|}{n}\right) R_i.$$
Observe that

$$\Lambda(\lambda) = \lim_{n\to\infty} \frac{1}{n}\Lambda_n(n\lambda) = \frac{P\lambda^2}{2}$$

and

$$\Lambda'(\lambda) = P\lambda.$$

By Gartner-Ellis theorem, we have $\hat{S}_n$ satisfies the LDP with good rate function

$$\Lambda^*(x) = \frac{x^2}{P} - \frac{x^2}{2P} = \frac{x^2}{2P}.$$