

C. FUNCTIONAL ANALYSIS

Theorem C.1 (Hahn-Banach Extension Theorem). *Let \mathcal{X} be a real vector space. Suppose that*

- \mathcal{Y} is a linear subspace of \mathcal{X} ,
- $p : \mathcal{X} \rightarrow \mathbb{R}$ is a convex functional,
- $f : \mathcal{Y} \rightarrow \mathbb{R}$ is linear and $f \leq p|_{\mathcal{Y}}$.

Then, there exists a linear functional $g : \mathcal{X} \rightarrow \mathbb{R}$ such that $g|_{\mathcal{Y}} = f$ and $g \leq p$.

Proof. The theorem is proved by transfinite induction.

If $\mathcal{Y} \neq \mathcal{X}$, choose $y \in \mathcal{Y} \setminus \mathcal{X}$, $\mathcal{Y}' = (\text{span}\{y\} \cup \mathcal{X})$ and extend f to $f' : \mathcal{Y}' \rightarrow \mathbb{R}$ by

$$f'(x + ty) = f(x) + t\alpha \text{ for all } x \in \mathcal{Y}, t \in \mathbb{R},$$

where one may choose, if possible,

$$\alpha(= f'(x')) \in \left[\sup_{t < 0} \sup_{x \in \mathcal{Y}} \frac{p(x + tx') - f(x)}{t}, \inf_{t > 0} \inf_{x \in \mathcal{Y}} \frac{p(x + tx') - f(x)}{t} \right]$$

so that $f'(x) \leq p(x)$ on \mathcal{Y}' . To see the above interval is indeed nonempty, note that the convexity of p together with $p \geq f$ implies that for each $t > 0$,

$$\frac{p(z + tx') - f(z)}{t} \geq \frac{p(z' - tx') - f(z')}{-t} \text{ for all } z, z' \in \mathcal{Y}, t > 0,$$

and hence the non-emptiness follows from the finite intersection property.

To conclude the proof, consider the set \mathcal{C} of all pairs (\mathcal{Y}, f) such that \mathcal{Y} is a subspace of \mathcal{X} and that $f : \mathcal{Y} \rightarrow \mathbb{R}$ is linear on \mathcal{Y} with $f \leq p|_{\mathcal{Y}}$. Define further a partial order \leq by

$$(\mathcal{Y}, f) \leq (\mathcal{Y}', f') \text{ if and only if } \mathcal{Y} \subseteq \mathcal{Y}' \text{ and } f'|_{\mathcal{Y}} = f.$$

Hence, for any chain $\{(\mathcal{Y}_i, f_i)\} \subset \mathcal{C}$, the space $\mathcal{Y} = \cup_i \mathcal{Y}_i$ is a vector space and the function $f(x) := f_i(x)$ whenever $x \in \mathcal{Y}_i$, which form an maximal element $(\mathcal{Y}, f) \in \mathcal{C}$ of the chain. Hence, by Zorn's lemma, there exists a maximal element $(\mathcal{Y}, f) \in \mathcal{C}$, and $\mathcal{Y} = \mathcal{X}$ by the first part of the proof. \square

In the context of topological vector space, an equivalent of Theorem C.1 is phrased in terms of separation as in the following. Recall that a *core point* x of a subset A of a vector space \mathcal{X} is the point satisfying that for every $y \subset \mathcal{X}$ there exists $\varepsilon = \varepsilon(y)$ such that $y + \delta y \in A$ for all $|\delta| \leq \varepsilon$, for which we denote $\text{cor}A = \{x \in A : x \text{ is a core point}\}$.

Theorem C.2 (Hahn-Banach Separation Theorem). *Let A and B be disjoint, nonempty, convex subsets of a real topological vector space \mathcal{X} . Assume that $\text{cor}A \neq \emptyset$. Then there is a linear functional $f : \mathcal{X} \rightarrow \mathbb{R}$ satisfying $\sup_{a \in A} f(a) \leq \inf_{b \in B} f(b)$ and $A \cup B \not\subset \{f = \alpha\}$ for all α . Moreover, if in addition $\text{int}A \neq \emptyset$, then f can be chosen so that $f \in \mathcal{X}^*$ and $f(a) < f(b)$ for all $a \in A$ and $b \in B$.*

Proof of Theorem C.2 by Theorem C.1. We first prove the general case, after which we point out the necessary modifications for the case where $\text{int}A \neq \emptyset$.

Let $C = A - B$ and $z \in \text{cor}(A - B)$. Define $C = A - B - z$ so that $-z \notin C$ and its associated *gauge*

$$p_C(x) = \inf\{t > 0 : x \in tC\}.$$

Then, $p_C : \mathcal{X} \rightarrow \mathbb{R}$ is a sublinear functional. Indeed, since $0 \in \text{cor}C$, p_C is real-valued. Moreover, $p_C(tx) = tp_C(x)$ for all $t > 0$ by definition and $p_C(x+y) \leq p_C(x) + p_C(y)$ by convexity of C . Consider the subspace $\mathcal{Y} = \text{span}\{-z\}$ and define the linear functional $g(-tz) = t$. To apply the extension theorem, observe that $p_C(-z) \geq 1$ and thus $g \leq p|_{\mathcal{Y}}$, yielding a linear extension $f : \mathcal{X} \rightarrow \mathbb{R}$ of g satisfying $f \leq p_C$ by Theorem C.1. To see f separates A and B , observe that for any $a \in A$ and $b \in B$,

$$\begin{aligned} f(a) &= f(a-b-z) + f(z) + f(b) \\ &\leq p_C(a-b-z) + f(z) + f(b) \leq 1 - 1 + f(b) = f(b). \end{aligned} \tag{C.1}$$

The number α in question can be arbitrary in $[\sup_{a \in A} f(a), \inf_{b \in B} f(b)]$. To see that $A \cup B \not\subset \{f = \alpha\}$, note that $z = a - b$ for some $a \in A$ and $b \in B$ and $f(z) = -1 = f(a) - f(b)$ as desired.

Now if $\text{int}A \neq \emptyset$, pick $z \in \text{int}(A - B)$ and proceed as before. Under the circumstances, C is a neighborhood of 0. Hence, $|f(x)| \leq \max\{|p_C(x)|, |p_C(-x)|\} \leq 1$ for all x in the neighborhood $C \cap -C$ of 0, proving the continuity. Furthermore, $p_C(a - b - z) < 1$ due to the fact $z \in \text{int}(A - B)$, rendering (C.1) a strict inequality. \square

Proof of Theorem C.1 by Theorem C.2. Let

$$A = \{(x, \alpha) \in \mathcal{X} \times \mathbb{R} : p(x) < \alpha\} \text{ and } B = \{(y, \beta) \in \mathcal{Y} \times \mathbb{R} : f(y) \geq \beta\}.$$

Then, A and B are convex and every point in A is a core point. Thus, by Theorem C.2, there exists a linear functional $h : \mathcal{X} \rightarrow \mathbb{R}$ and numbers $\rho \in \mathbb{R}$ such that

$$\sup_{(x, \alpha) \in A} h(x) + \rho\alpha \leq \gamma := \inf_{(y, \beta) \in B} h(y) + \rho\beta \quad (\gamma \in \mathbb{R}). \tag{C.2}$$

Under the circumstances, we observe, by letting $\alpha \rightarrow \infty$, that $\rho \leq 0$. It is clear that $\rho \neq 0$, for if otherwise, the inequality (C.1) above holds if and only if $h(x) = 0$ for all $x \in \mathcal{X}$, contradicting Theorem C.2 as $h(A) = h(B) = \{0\}$. Now that $\rho < 0$, the inequality (C.2) implies that

$$\begin{aligned} \rho^{-1}h(y) + f(y) &\leq \rho^{-1}\gamma \text{ for all } y \in \mathcal{Y} \quad \Rightarrow \quad f = \rho^{-1}(\gamma - h|_{\mathcal{Y}}), \\ \rho^{-1}g(x) + p(x) &\geq \rho^{-1}\gamma \text{ for all } x \in \mathcal{X} \quad \Rightarrow \quad p \geq \rho^{-1}(\gamma - h). \end{aligned}$$

The proof is finished by choosing $g = \rho^{-1}(\gamma - h)$. \square

Theorem C.3. *Suppose \mathcal{X} is a locally convex real topological vector space. Suppose A and B are two disjoint, nonempty, convex sets in the locally convex topological vector space \mathcal{X} . If A is compact and B is closed, then there exists an $f \in \mathcal{X}^*$ such that $\sup_{a \in A} f(a) < \inf_{b \in B} f(b)$.*

Proof. Suppose V is a convex neighborhood of 0 such that $(A + V) \cap B = \emptyset$. Apply Theorem C.2 to $A + V$ and B to obtain $f \in \mathcal{X}^*$ such that $\sup_{a \in A} f(a) < \sup_{a \in A+V} f(a)$ since $f(A)$ is compact and $f(A + V)$ is open. \square