4. Some backgrounds

The general setting for the rest of this note is that $\mathcal{X}$ is, unless specified otherwise, a regular Hausdorff topological space with $\mathcal{B}_{\mathcal{X}} \subset \mathcal{B}$, as defined below.

Definition 4.1 (Hausdorff topological space)**.** A topological space $\mathcal{X}$ is said to be *regular* if for every point x and closed set y there exists disjoint open sets U, V satisfying $x \in U$ and $y \in V$.

Definition 4.2 (regular topological space)**.** A topological space $\mathcal{X}$ is said to be *regular* if for every point x and closed set $F \not\supseteq \{x\}$ there exists disjoint open sets U, V satisfying $x \in U$ and $F \subset V$.

4.1. Existence and uniqueness of LDP.

Proposition 4.3 (uniqueness)**.** *A family of probability measures $\{\mu_\varepsilon\}$ on a regular topological space can have at most one rate function associated with its LDP.*

Proof. Suppose I_1 and I_2 are two rate functions associated with the LDP of μ_ε, with a point x_0 satisfying $I_1(x_0) < I_2(x_0)$. Then, by lower semicontinuity, there exists an open neighborhood A of x_0 such that $\inf_{x\in A} I_2(x) \geq I_2^\delta(x_0)$, where I_2^δ is the δ-rate function as defined in Definition 3.8. By regularity, we may assume $\inf_{x\in\overline{A}} I_2(x) \geq I_2^\delta(x_0)$ (by choosing a subset if necessary). Hence,

$$-I_2^\delta(x_0) \geq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geq \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geq -I_1(x_0),$$

contradicting $I_1(x_0) < I_2(x_0)$ if letting $\delta \to 0$. □

Remark 4.4.

(1) *Regular spaces include all metric spaces.*
(2) *If $\mathcal{X}$ is regular and locally compact, then the LDP Proposition 4.3 can be relaxed by weak LDP.*

Proposition 4.5 (existence)**.** *Let $\mathcal{A}$ be a base of the topology of the topological space $\mathcal{X}$. For every $A \in \mathcal{A}$, define*

$$\underline{\mathcal{L}}_A := \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \quad \text{and} \quad \overline{\mathcal{L}}_A := \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A),$$

and

$$\underline{I}(x) := -\inf_{A\in\mathcal{A}: x\in A} \underline{\mathcal{L}}_A \quad \text{and} \quad \overline{I}(x) := -\inf_{A\in\mathcal{A}: x\in A} \overline{\mathcal{L}}_A.$$

If $\underline{I} = \overline{I}$, then μ_ε satisfies the weak LDP with the rate function $\underline{I}$.

Proof. We begin by noting the fact that I is indeed a rate function, which follows naturally from definition.

For the lower bound in LDP, observe that for every open set G and any $x \in G$, there exists $A \in \mathcal{A}$ such that $x \in A \subset G$ such that

$$\underline{\mathcal{L}}_G \geq \underline{\mathcal{L}}_A \geq -\underline{I}(x),$$

from which the desired lower bound follows naturally.

For upper bound, given any compact set F, we can find for every point x a neighborhood A_x satisfying $\overline{\mathcal{L}}_{A_x} \leq -\overline{I}^\delta(x)$. By compactness, it admits some finite open subcover $\{A_{x_i}\}_{i=1}^m$, and thus

$$\overline{\mathcal{L}}_F \le \max_{1\le i\le m} \overline{\mathcal{L}}_{A_i} \le \max_{1\le i\le m} -\overline{I}^\delta(x_i) \le -\inf_{x\in F} \overline{I}^\delta(x).$$

This finishes the proof. □

Remark 4.6. *Proposition 4.5 holds for parametrized family* $\{\mu_{\varepsilon,\sigma}\}$ *for any given* $\sigma \in \Sigma$ *if*

$$\underline{\mathcal{L}}_A := \liminf_{\varepsilon\to 0} \varepsilon \log\Big[\inf_{\sigma\in\Sigma} \mu_{\varepsilon,\sigma}(A)\Big] \quad \text{and} \quad \overline{\mathcal{L}}_A := \limsup_{\varepsilon\to 0} \varepsilon \log\Big[\sup_{\sigma\in\Sigma} \mu_{\varepsilon,\sigma}(A)\Big].$$

This may be useful in describing certain processes, e.g., Markov chains conditioned on initial state σ*.*

We also have a partial converse of Proposition 4.5.

Proposition 4.7. *Suppose that* $\{\mu_\varepsilon\}$ *satisfies the LDP in a regular topological space* $\mathcal{X}$ *with rate function* I*. Then, for any base* $\mathcal{A}$ *of the topology of* $\mathcal{X}$*,* $I = \underline{I} = \overline{I}$ *with both functions defined in Proposition 4.5.*

Proof. Exercise. □