A. Sion's minimax theorem

The following proofs are excerpted from [1].

Definition A.1. Let V be a topological vector space. A function $f : V \to \mathbb{R}$ is said to be quasi-convex (respectively, quasi-concave) if $f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$ (respectively, $f(tx + (1-t)y) \geq \min\{f(x), f(y)\}$) for all $t \in [0,1]$.

Theorem A.2 (Sion's minimax theorem)**.** *Let X be a compact convex set in a topological vector space and Y is a convex set in a topological vector space. If $f : X \times Y \to \mathbb{R}$ is a function such that*

- *$f(x,\cdot)$ is upper semicontinuous and quasi-concave for all $x \in X$, and that*
- *$f(\cdot,y)$ is lower semicontinuous and quasi-convex for all $y \in Y$,*

then

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y)$$

Lemma A.3. *Suppose the assumptions of Theorem A.2 hold. If*

$$\min_{x\in X}\max\{f(x,y_1), f(x,y_2)\} > \alpha$$

for some $\alpha \in \mathbb{R}$ and $y_1, y_2 \in Y$, then there exists $y_0 \in Y$ such that $\min_{x\in X} f(x,y_0) > \alpha$.

Proof. Write $[y_1,y_2] = \{ty_1 + (1-t)y_2 : t \in [0,1]\}$ for short.

Suppose toward a contradiction that

$$\min_{x\in X} f(x,y) \leq \alpha < \beta < \min_{x\in X}\max\{f(x,y_1), f(x,y_2)\} \quad \text{for all} \quad y \in Y \tag{A.1}$$

Define the superlevel set

$$C_y^\gamma := \{x \in X : f(x,y) \leq \gamma\}$$

and observe that

- For all $y \in Y$, $C_y^\alpha \subseteq C_y^\beta$ are nonempty (by (A.1)), convex (by quasi-convexity) and compact (by lower semicontinuity).
- $C_{y_1}^\beta \cap C_{y_2}^\beta = \emptyset$ by (A.1) and $C_y^\beta \subset C_{y_1}^\beta \cup C_{y_2}^\beta$ for all $y \in [y_1,y_2]$ by quasi-concavity.
- For all $y \in Y$, $C_y^\alpha \subset C_y^\beta$ are convex by quasi-convexity and thus connected.

Inferring from the observations, we have either $C_y^\beta \subset C_{y_1}^\beta$ or $C_y^\beta \subset C_{y_2}^\beta$ for all $y \in [y_1,y_2]$. Furthermore, the sets

$$I = \left\{y \in [y_1,y_2] : C_y^\alpha \subset C_{y_1}^\beta\right\} \quad \text{and} \quad J = \left\{y \in [y_1,y_2] : C_y^\alpha \subset C_{y_2}^\beta\right\}$$

form a partition of $[y_1,y_2]$. It remains to show that both I and J are closed sets to reach a contradiction, as $[y_1,y_2]$ is connected. Note that $[y_1,y_2]$ is metrizable with the usual Euclidean distance. Hence, to demonstrate closedness one might examine whether the sequential limit z of a convergent sequence $(z_n)_{n=1}^\infty \subset I$ lies in I. Indeed, given any $x \in C_y^\alpha$

$$\alpha \geq f(x,z) \geq \limsup_{n\to\infty} f(x,z_n)$$

implying that

$$x \in C^{\beta}_{z_n} \subset C^{\beta}_{y_1}$$

for all sufficiently large n as desired. □

Lemma A.4. *Suppose the assumptions of Theorem A.2 hold. If*

$$\min_{x \in X} \max_{1 \le i \le n} f(x, y_i) > \alpha$$

for some $\alpha \in \mathbb{R}$ *and* $(y_i)_{i=1}^n \subset Y$*, then there exists* $y_0 \in Y$ *such that* $\min_{x \in X} f(x, y_0) > \alpha$*.*

Proof. We prove the lemma by induction. The case $n = 1$ is trivial. For induction step, we assume the hypothesis holds for $n - 1$ and define

$$X' = \{x \in X : f(x, y_n) \le \alpha\},$$

which is a convex set due to quasi-convexity. To proceed, we assume X' is nonempty, for the proof is complete otherwise. Under the circumstances,

$$\min_{x \in X'} \max_{1 \le i < n} f(x, y_i) > \alpha$$

and the induction hypothesis applies to yield some $y' \in Y$ satisfying

$$\min_{x \in X'} f(x, y') > \alpha.$$

Now that

$$\min_{x \in X} \max\{f(x, y'), f(x, y_n)\} > \alpha,$$

we once again apply Lemma A.3 to obtain the desired y_0. □

Proof of Theorem A.2. Let $\alpha < \min_{x \in X} \max_{y \in Y} f(x, y)$ and C^{α}_y as in Lemma A.3. Then, C^{α}_y is compact and $\cap_{y \in Y} C^{\beta}_y = \emptyset$. By the finite intersection property, there exists finite points $(y_i)_{i=1}^n$ such that $\cap_{i=1}^n C^{\beta}_{y_i} = \emptyset$ and therefore

$$\alpha < \min_{x \in X} \max_{1 \le i \le n} f(x, y_i).$$

By Lemma A.4, there exists $y_0 \in Y$ such that

$$\alpha < \min_{x \in X} f(x, y_0) \le \sup_{y \in Y} \min_{x \in X} f(x, y).$$

Immediately, this gives

$$\min_{x \in X} \sup_{y \in Y} f(x, y) \le \sup_{y \in Y} \min_{x \in X} f(x, y).$$

The remaining inequality simply follows from definition. □