TOPOLOGICALLY MIXING PROPERTIES OF MULTIPLICATIVE INTEGER SYSTEM

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Abstract. Motivated from the study of multiple ergodic average, the investigation of multiplicative shift spaces has drawn much of interest among researchers. This paper focuses on the relation of topologically mixing properties between multiplicative shift spaces and traditional shift spaces. Suppose that $\mathsf{X}_{\Omega}^{(\ell)}$ is the multiplicative subshift derived from the shift space Ω with given $\ell > 1$. We show that $\mathsf{X}_{\Omega}^{(\ell)}$ is (topologically) transitive/mixing if and only if Ω is extensible/mixing. After introducing ℓ-directional mixing property, we derive the equivalence between ℓ-directional mixing property of $\mathsf{X}_{\Omega}^{(\ell)}$ and weakly mixing property of Ω.

1. Introduction

Let $\mathcal{A} = \{0, 1, \dots, m-1\}$ be a finite alphabet and let $\Omega \subseteq \mathcal{A}^{\mathbb{N}}$ be a shift space with the shift map $\sigma : \Omega \to \Omega$ defined by $(\sigma x)_i = x_{i+1}$ for $i \in \mathbb{N}$. Suppose $1 < \ell$ is a natural number. Kenyon et al. [11] defined the *multiplicative subshift* $\mathsf{X}_{\Omega}^{(\ell)} \subseteq \mathcal{A}^{\mathbb{N}}$ as

$$\mathsf{X}_{\Omega}^{(\ell)} = \{x = (x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : (x_{i\ell^{n-1}})_{n \in \mathbb{N}} \in \Omega \text{ for all } i\}. \tag{1}$$

The name "*multiplicative subshift*" follows from the fact $\mathsf{X}_{\Omega}^{(\ell)}$ is multiplicatively invariant in the sense $\Pi_q \mathsf{X}_{\Omega}^{(\ell)} \subseteq \mathsf{X}_{\Omega}^{(\ell)}$ for all $q \in \mathbb{N}$, where $\Pi_q x = (x_{qk})_{k=1}^{\infty}$. The study of (1) takes its origin from the multifractal analysis of the 0-level set $E_{\Phi}(0)$ considered by Fan et al. [8], where

$$E_{\Phi}(\theta) = \{x \in \{0,1\}^{\mathbb{N}} : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} x_k x_{2k} = \theta\}. \tag{2}$$

The study of (2) is a special case of the multiple ergodic averages $A_n \Phi(x) := \frac{1}{n} \sum_{k=0}^{n-1} \Phi(T_1^k x, \dots, T_d^k x)$ with $d = 2$, $T_i = \sigma^i$, and $\Phi(x, y) = x_1 y_1$. In the same paper, they also studied the subset

$$\mathsf{X}_2 = \{x \in \{0,1\}^{\mathbb{N}} : x_i x_{2i} = 0 \text{ for all } i\} \tag{3}$$

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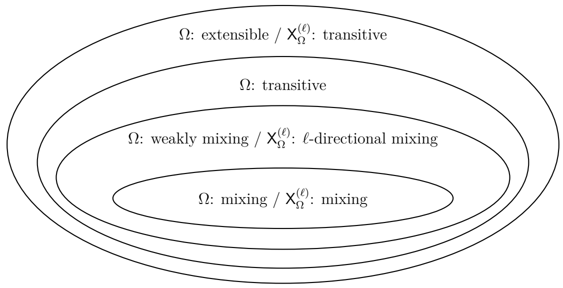


FIGURE 1. Summary of the result

of (2). It is easily seen that X_2 is a special type of $\mathsf{X}_\Omega^{(\ell)}$ in which Ω is the *golden mean shift* (it is called "*multiplicative golden mean shift*" in [11]). Moreover, the box dimensions of X_2 is $\dim_B \mathsf{X}_2 = \frac{1}{2\log 2}\sum_{n=1}^{\infty} \frac{\log F_n}{2^n}$, where F_n is the Fibonacci sequence: $F_0 = 1$, $F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Later, Kenyon et al. [11] obtained the general formula of Hausdorff and box dimensions of X_Ω. It is known that the subshift corresponding to the closed invariant subsets of $[0,1]$ under the map $x \mapsto mx$ (mod 1) has the property that the Hausdorff and box dimensions are coincident [10]. Even though the Hausdorff and box dimension of X_Ω are not coincident generally (the Hausdorff dimension is less than or equal to the box dimension), the characterization of the equality is addressed therein. Furthermore, Peres and Solomyak [15] gave the full dimension spectrum of $\dim_H E_\Phi(\theta)$, and mentioned that $\dim_H \mathsf{X}_2 = \dim_H E_\Phi(0)$. Beyond X_2, more dimension results can be found. Peres et al. [14] considered the multiplicative subshifts $\mathsf{X}_\Omega^{(S)}$, for which S is the semigroup generated by primes $p_1, \ldots, p_k$. Namely,

$$\mathsf{X}_\Omega^{(S)} := \{x = (x_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} : x|_{iS} \in \Omega \text{ for all } i \text{ such that } (i,S) = 1\}.$$

A typical example of $\mathsf{X}_\Omega^{(S)}$ is

$$\mathsf{X}_{2,3} = \{x \in \{0,1\}^{\mathbb{N}} : x_k x_{2k} x_{3k} = 0 \text{ for all } k\},$$

where S is the semigroup generated by 2 and 3. The authors in [14] extended [11] to $\mathsf{X}_\Omega^{(S)}$ and obtained the Hausdorff and box dimensions of $\mathsf{X}_\Omega^{(S)}$. Ban et al. [3] approximated the box dimension $\dim_B \left(\mathsf{X}_\Omega^{(S)} \cap \Omega\right)$ for the case where Ω is a subshift of finite type. Fan et al. [9] gave a complete solution to the problem of multifractal analysis of the limit of the multiple ergodic averages $\frac{1}{n}\sum_{i=1}^{n} \phi(x_i, x_{i\ell}, \ldots, x_{i\ell^{k-1}})$ for $k, \ell \geq 2$. We refer to [7] for a nice state-of-the-art survey of the the multiple ergodic averages.

The dimensional aspect of the multiplicative subshifts exhibits that the topological behaviors of $\mathsf{X}_\Omega^{(\ell)}$ or $\mathsf{X}_\Omega^{(S)}$ are multidimensional rather than one-dimensional, which is out of the blue since multiplicative subshifts come from one-dimensional shift spaces. Such a transformation comes from the multiplicative action on the shift spaces (cf. [3, 8, 9, 11, 14, 15]). As different types of mixing properties offer different phenomena such as positive topological entropy, density of periodic points, and the domino problem, it is essential to study which mixing properties a given multiplicative shift satisfies (cf. [5, 6, 13, 16, 17] for shift spaces over $\mathbb{Z}^d$ and [1, 2] for shift spaces over free monoids). This paper aims to connect the mixing properties of Ω and $\mathsf{X}_\Omega^{(\ell)}$. To be more specific, assuming Ω possesses some topological property $\mathbf{P}$, can we say something about the mixing properties of $\mathsf{X}_\Omega^{(\ell)}$? Are the properties $\mathsf{X}_\Omega^{(\ell)}$ equipped with stronger or weaker than property $\mathbf{P}$? In other words, the goal of this paper is trying bridge the topological behaviors of the two spaces: one is additively invariant and the other is multiplicatively invariant. The main contribution of this paper is Theorems 1.1, 1.2, and 1.3 (cf. Figure 1), which explicitly characterizes different types of mixing properties for multiplicative shifts that is a difficult task for multidimensional shift spaces. Particularly, the notion of directional mixing is closely related to the notion of weakly mixing, which, intuitively, reveals that the specific shift is mixing in some direction. In the following, we denote by $B_n(\mathsf{X}_\Omega^{(\ell)})$ the set of all the finite prefixes u whose supports $s(u) = [1, n]$ and write $B(\mathsf{X}_\Omega^{(\ell)}) = \cup_{n\geq 1} B_n(\mathsf{X}_\Omega^{(\ell)})$. First theorem demonstrates the connection between transitivity (for $\mathsf{X}_\Omega^{(\ell)}$) and extensibility (for Ω).

Theorem 1.1. *The following are equivalent.*

(1) Ω *is extensible.*
(2) $\mathsf{X}_\Omega^{(\ell)}$ *is transitive.*
(3) For $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ *there exists* $\alpha \in \mathbb{N} \setminus \ell\mathbb{N}$ *such that for any* $k \in \mathbb{N}_0$ *there exists* $x \in \mathsf{X}_\Omega^{(\ell)}$ *such that* $x|_{s(u)} = u$ *and* $(\Pi_{|u|\alpha\ell^k} x)|_{s(v)} = v$.
(4) $\mathsf{X}_\Omega^{(\ell)}$ *is* L*-directional mixing for some* L *which has a prime factor* $p \in \mathbb{P}$ *satisfying* $p \nmid \ell$.
(5) $\mathsf{X}_\Omega^{(\ell)}$ *is* L*-directional mixing for every* L *which has a prime factor* $p \in \mathbb{P}$ *satisfying* $p \nmid \ell$.

The second theorem reveals that Ω being weakly mixing is an equivalent condition of ℓ-directional mixing of the associated multiplicative shift.

Theorem 1.2. *The following are equivalent.*

(1) Ω *is weakly mixing.*
(2) $\mathsf{X}_\Omega^{(\ell)}$ *is* ℓ*-directional mixing.*
(3) $\mathsf{X}_\Omega^{(\ell)}$ *is* ℓ^n*-directional mixing for every* $n \in \mathbb{N}$.
(4) $\mathsf{X}_\Omega^{(\ell)}$ *is* ℓ^n*-directional mixing for some* $n \in \mathbb{N}$.

Furthermore, if ℓ *is a prime number, the following is equivalent as above.*

(5) $\mathsf{X}_\Omega^{(\ell)}$ *is* L*-directional mixing for every* $L > 1$.

The last main theorem of this paper, which is rather not that out of the blue but is still showing there is more to be investigated, indicates that Ω is mixing if and only if $\mathsf{X}_\Omega^{(\ell)}$ is mixing.

Theorem 1.3. *The following are equivalent.*

(1) Ω *is mixing.*
(2) $\mathsf{X}_{\Omega}^{(\ell)}$ *is mixing.*
(3) *For* $u, v \in B(\mathsf{X}_{\Omega}^{(\ell)})$*, there exists* $N \in \mathbb{N}$ *such that for* $k \geq N$ *and* $\alpha \in \mathbb{N} \setminus \ell\mathbb{N}$ *there exists* $x \in \mathsf{X}_{\Omega}^{(\ell)}$ *such that* $x|_{s(u)} = u$ *and* $(\Pi_{|u|\alpha\ell^k} x)|_{s(v)} = v$*.*

We organize the material of this paper as follows. Section 2 elucidates the definitions and propositions that are used in this investigation and gives some examples to illustrate the idea of the proof of the main theorems. Section 3 is devoted to the proofs of Theorems 1.1, 1.2, and 1.3. Discussions and some open problems are carried out in Section 4.

2. Definitions and Examples

In this section, we recall some fundamental definitions and facts of mixing properties for dynamical systems, and then address them to the case of multiplicative shifts in an equivalent way. The definitions of mixing properties on abstract dynamical systems are known as follows.

Definition 2.1. An action $\pi : M \curvearrowright X$ of a monoid M on a topological space X by continuous maps is

(1) *extensible* if for all nonempty open set $U \subset X$ and finite set $F \subset M$ there exists $m \in M \setminus F$ with $U \cap \pi_m^{-1} X \neq \varnothing$,
(2) *transitive* if for all nonempty open sets $U, V \subset X$ there exists $m \in M$ with $U \cap \pi_m^{-1} V \neq \varnothing$,
(3) *weakly mixing* if the diagonal action $M \curvearrowright X \times X$ is transitive,
(4) *mixing* if for all nonempty open sets $U, V \subset X$ there exists $F \subset M$ finite such that $U \cap \pi_m^{-1} V \neq \varnothing$ for all $m \in M \setminus F$,

where $\pi_m(\cdot) := \pi(m, \cdot)$.

Let $\mathcal{A}$ be a finite alphabet and let $\Omega \subset \mathcal{A}^{\mathbb{N}}$ be a shift space with the shift map $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$. That is, the shift map is an action $\mathbb{N} \curvearrowright \Omega$ of $\mathbb{N}$ on Ω. Under the prodiscrete topology, the consideration of open sets of Ω focuses on the cylinder sets without loss of generality. Denote the set of all admissible words of length n by $B_n(\Omega)$ and set $B(\Omega) = \cup_{n \geq 1} B_n(\Omega)$. For every $u \in B(\Omega)$ and $x \in \Omega$, let $|u|$ be the length of u and $x_{[i,j]} = (x_i, \ldots, x_j)$ be the *projection* of x on $[i, j] := \{n \in \mathbb{N} : i \leq n \leq j\}$. In this particular case, the definition above can be translated into the following equivalent statements.

Definition 2.2. Let (Ω, σ) be a shift space. We say that (Ω, σ) is

(1) *extensible* if for all $u \in B_n(\Omega)$ and for all $m \in \mathbb{N}$, there exists $x \in \Omega$ such that $x_{[m+1,m+n]} = u$;

(2) *transitive* if for all $u \in B_n(\Omega), v \in B_k(\Omega)$ there exists $m \in \mathbb{N}$ and $x \in \Omega$ such that $x_{[1,n]} = u$ and $x_{[n+m+1,n+m+k]} = v$;
(3) *totally transitive* if σ^n is transitive for all $n \geq 1$;
(4) *weakly mixing* if for all $u_i \in B_{n_i}(\Omega), v_i \in B_{k_i}(\Omega)$, $i = 1, 2$, there exist $m \in \mathbb{N}$ and configurations $x^{(i)} \in \Omega$ such that $x^{(i)}|_{[1,n_i]} = u_i$ and $x^{(i)}|_{[n_i+m+1,n_i+m+k_i]} = v_i$;
(5) *mixing* if for all $u \in B_n(\Omega), v \in B_k(\Omega)$ there exists $N \in \mathbb{N}$ such that for $m \geq N$ there exists $x \in \Omega$ such that $x_{[1,n]} = u$ and $x_{[n+m+1,n+m+k]} = v$.

It follows from the definitions that

$$\text{mixing} \Rightarrow \text{weakly mixing} \Rightarrow \text{totally transitive} \Rightarrow \text{transitive} \Rightarrow \text{extensible}.$$

Roughly speaking, a weakly mixing system transits any pair of open sets to another pair simultaneously. Furstenberg [10] demonstrated the property holds for finitely many sets.

Proposition 2.3 (See [10])**.** *Suppose Ω is a shift space. Then following statements are equivalent.*

(1) Ω is weakly mixing.
(2) For $\{u_1, u_2, \ldots, u_M\}, \{v_1, v_2, \ldots, v_M\} \subset B(\Omega)$ there exist $m \in \mathbb{N}$ and configurations $\{x^{(i)}\}_{i=1}^M$ satisfying $x^{(i)}|_{s(u_i)} = u_i$ and $(\sigma^m x^{(i)})|_{s(v_i)} = v_i$ for $1 \leq i \leq M$.

For each shift space Ω and natural number $\ell \geq 2$, the multiplicative shift space $\mathsf{X}_\Omega^{(\ell)}$ is defined as

$$\mathsf{X}_\Omega^{(\ell)} = \{x = (x_k)_{k=1}^\infty \in \mathcal{A}^\mathbb{N} : (x_{i\ell^{n-1}})_{n\in\mathbb{N}} \in \Omega \text{ for all } i\}.$$

We say that $u \in \mathcal{A}^S$ is a *pattern* in $\mathsf{X}_\Omega^{(\ell)}$ if there exist a finite set $S \subset \mathbb{N}$ and $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_S = u$, i.e., $x_i = u_i$ for $i \in S$. In this case, S is called the *support* of u and is denoted by $s(u)$. The *multiplicative action* $\Pi : \mathbb{N} \curvearrowright \mathsf{X}_\Omega^{(\ell)}$ is defined as

$$(\Pi_q x)_i := x_{qi} \quad \text{for} \quad i \in \mathbb{N}, x \in \mathsf{X}_\Omega^{(\ell)}.$$

It is obvious that $\mathsf{X}_\Omega^{(\ell)}$ is invariant under the multiplicative map and thus $(\mathsf{X}_\Omega^{(\ell)}, \Pi)$ is a dynamical system. Therefore, the discussion of topological properties in Definition 2.2 is of interest. Observe that $\Pi_m U \subseteq U$ for each open set $U \subseteq \mathsf{X}_\Omega^{(\ell)}$ and $m \in \mathbb{N}$ due to the prodiscrete topology on $\mathcal{A}^\mathbb{N}$. It can be verified without difficulty that the transitivity and mixing property of the multiplicative action Π on $\mathsf{X}_\Omega^{(\ell)}$ are translated in the following equivalent statements, where an auxiliary multiplier $|u|$ is introduced in for avoidance of the overlaps between the supports of u and v.

Proposition 2.4. *A multiplicative shift $(\mathsf{X}_\Omega^{(\ell)}, \Pi)$ is*

(1) transitive *if and only if for $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ there exist $m \in \mathbb{N}$ and $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_{s(u)} = u$ and $\left(\Pi_{|u|m} x\right)|_{s(v)} = v$;*
(2) mixing *if and only if for $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ there exists $N \in \mathbb{N}$ such that for $m \geq N$ there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x_{s(u)} = u$ and $\left(\Pi_{|u|m} x\right)|_{s(v)} = v$.*

The idea of these definitions is to connect the patterns u, v in $\mathsf{X}_\Omega^{(\ell)}$ under the action of multiplicative semigroup of positive integers. Observe that, for each given integer $\ell \geq 2$, every natural number n has a unique decomposition $n = \alpha\ell^k$, where $\alpha \in \mathbb{N} \setminus \ell\mathbb{N}$ and $k \geq 0$. The following proposition comes from straightforward examination, and thus the proof is omitted.

Proposition 2.5. *Consider the multiplicative shift* $(\mathsf{X}_\Omega^{(\ell)}, \{\Pi_q\}_{q\in\mathbb{N}})$. *Then* $\mathsf{X}_\Omega^{(\ell)}$ *is*

(1) transitive *if and only if for* $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ *there exists* $(\alpha, k) \in (\mathbb{N} \setminus \ell\mathbb{N}) \times \mathbb{N}_0$ *and* $x \in \mathsf{X}_\Omega^{(\ell)}$ *such that* $x|_{s(u)} = u$ *and* $\left(\Pi_{|u|\alpha\ell^k} x\right)|_{s(v)} = v$*;*

(2) mixing *if and only if for all* $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ *there exists* $N \in \mathbb{N}$ *such that if* $(\alpha, k) \in (\mathbb{N} \setminus \ell\mathbb{N}) \times \mathbb{N}_0$ *with* $\alpha\ell^k \geq N$ *there exists* $x \in \mathsf{X}_\Omega^{(\ell)}$ *such that* $x|_{s(u)} = u$ *and* $\left(\Pi_{|u|\alpha\ell^k} x\right)|_{s(v)} = v$.

It is note worthy that the definition of multiplicative map Π can be extended to $B(\mathsf{X}_\Omega^{(\ell)})$ as follows: For each $q \in \mathbb{N}$, the multiplicative map $\Pi_q : \cup_{n\in\mathbb{N}}\mathcal{A}^{[1,n]} \to \cup_{n\in\mathbb{N}}\mathcal{A}^{[1,n]}$ is defined as $(\Pi_q u)_i = u_{qi}$ for $i \in [1, \lfloor\frac{|u|}{q}\rfloor]$, where $[1, \lfloor\frac{|u|}{q}\rfloor]$ is the support of $\Pi_q u$.

For a multiplicative shift $\mathsf{X}_\Omega^{(\ell)}$ and $m, n \in \mathbb{N}$, we define $m \simeq n$ if and only if $\dfrac{m}{n} = \ell^i$ for some $i \in \mathbb{Z}$. It follows immediately that $\simeq$ is an equivalence relation. With this we define $\Lambda_{[i]} = \{i\ell^k : k \in \mathbb{Z}\} \cap \mathbb{N}$ for each $i \in \mathbb{N}$, where $[i]$ denotes the equivalence class of i with respect to $\simeq$. The following proposition comes straightforwardly from the fundamental theorem of arithmetic.

Proposition 2.6. *Let* $\Lambda_{[i]}$ *be defined as above. Then*

(1) $\{\Lambda_{[i]}\}_{i\in\mathbb{N}}$ *is a partition of* $\mathbb{N}$*;*

(2) for each $\alpha \in \mathbb{N} \setminus \ell\mathbb{N}$, $\alpha\Lambda_{[i]} \subset \Lambda_{[j]}$ *for some* $[j] \neq [i]$. *In particular,* $\alpha\Lambda_{[i]} = \Lambda_{[\alpha i]}$ *if* $\ell \in \mathbb{P}$, *where* $\mathbb{P}$ *denotes the set consisting of all prime numbers.*

For the sake of simplicity, we refer to $\Lambda_{[i]}$ as Λ_i with an extra requirement that i is the smallest element of $[i]$ for the rest of this paper unless otherwise stated. For every $u \in B(\mathsf{X}_\Omega^{(\ell)})$, we define $u|_{\Lambda_i} \in B(\Omega)$ and $\xi(u)$ as $u|_{\Lambda_i} := (u_{i\ell^{j-1}})_j$ for $1 \leq j \leq \lfloor \log_\ell \frac{|u|}{i} \rfloor$ and $\xi(u) := \max\{n : n \leq |u|, \ell \nmid n\}$, respectively. In addition, for $1 < q, N \in \mathbb{N}$, let $A_q := \mathbb{N} \setminus q\mathbb{N}$ denote the set of positive integers which are not divisible by q and $A_{q,N} := A_q \cap [1, N]$.

Example 2.7. Suppose $\mathcal{A} = \{0, 1\}$, $\ell = 2$, and Ω is the one-sided golden mean shift. Then

$$\mathsf{X}_2 := \mathsf{X}_\Omega^{(2)} = \mathcal{A}^{\mathbb{N}}.$$

If $u \in \mathcal{A}^{[1,8]}$ and $v \in \mathcal{A}^{96\cdot[1,8]}$, then $\xi(u) = 7$, which means that $s(u) \cap \Lambda_7 \neq \varnothing$. The support of u and is illustrated in Figure 2. Note that in the figure, the elements of the monoid $\mathbb{N}$ is rearranged to sit in the $\mathbb{N}^2$ lattice in the following manner: If the set $A_\ell = \{A_\ell^{(1)} < A_\ell^{(2)} < \ldots\}$ is endowed with the natural order of integers, every integer $A_\ell^{(j)} \cdot \ell^{i-1} \in \mathbb{N}$ has the coordinate (i, j) in the lattice and thus the set of

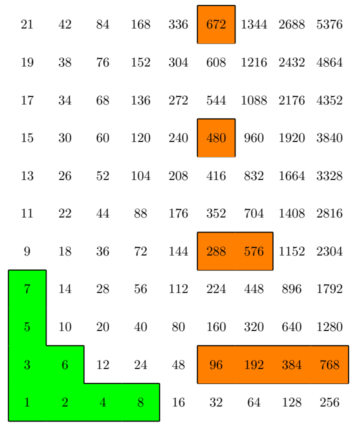


FIGURE 2. The multiplicative map Π breaks the topological structure of the support of pattern in $\mathsf{X}_\Omega^{(2)}$. The set $[1,8]$ is the green region, while the set $96 \cdot [1,8]$ is colored in orange. Notably, $[1,8]$ is connected while $96 \cdot [1,8]$ is broken in the $\mathbb{N}^2$-lattice partitioned by $\{\Lambda_i\}_{2\nmid i}$.

all elements in the j-th is exactly $\Lambda_{A_\ell^{(j)}}$. The supports of u and v are colored in green and orange, respectively. Observe that the multiplicative map Π breaks the topological structure of the support of pattern. In general, for any $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ and $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $(\Pi_{|u|\alpha\ell^k} x)|_{s(v)} = v$ and $x|_{s(u)} = u$, the supports of u and $x|_{|u|\alpha\ell^k s(v)}$ are non-overlapping if $\alpha\ell^k \neq 1$.

One of the main difference between traditional shift spaces and multiplicative shift spaces is that the multiplicative map messes up the topological structure of the underlying space, which makes the investigation of dynamical phenomena of multiplicative shift spaces much more complicated and diversifies mixing properties in multiplicative subshifts. The following definition introduces a mixing property called *directional mixing* that is related to the weakly mixing property in traditional shift spaces.

Definition 2.8. Suppose $\mathsf{X}_\Omega^{(\ell)}$ is a multiplicative shift space. We say that $\mathsf{X}_\Omega^{(\ell)}$ is q-*directional mixing* for some $q > 1$ if for $u, v \in B(\mathsf{X}_\Omega^{(\ell)})$ there exists $k \in \mathbb{N}_0$ such that for any $\alpha \in A_q$ there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ satisfying $x|_{s(u)} = u$ and $(\Pi_{|u|\alpha q^k} x)|_{s(v)} = v$.

At this point, it is seen through the following example that the transitivity and directional mixing properties are actually not equivalent.

Example 2.9. The following example distinguishes Theorems 1.1 and 1.2. Let $\Omega = \mathsf{X}_{\mathcal{F}}$ be defined by forbidden set $\mathcal{F} = \{01\}$. Then, Ω is extensible yet not transitive, and $\mathsf{X}_\Omega^{(\ell)}$ satisfies Theorem 1.1. However, $\mathsf{X}_\Omega^{(\ell)}$ is not ℓ-directional mixing. It can be verified by considering $u' = 0^\ell, v' = 1 \in B(\mathsf{X}_\Omega^{(\ell)})$ and $\alpha = 1$. Under the

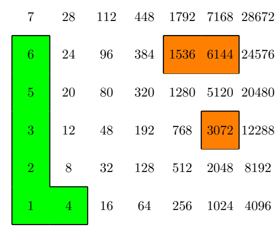


FIGURE 3. The weakly mixing property of Ω implies $\mathsf{X}_\Omega^{(\ell)}$ is ℓ-directional mixing. When $\alpha \cdot \ell^k = 1 \cdot 4^4$, the (partial) supports of u' and $x|_{|u'|\alpha\ell^k s(v')}$ are colored in green and orange, respectively. See Example 2.11 for more details.

circumstances, no $y \in \mathsf{X}_\Omega^{(\ell)}$ should accept v' since 1 at the the position where v' is located is to the right of 0.

Theorem 1.1 reveals that Ω is extensible if and only if $\mathsf{X}_\Omega^{(\ell)}$ is transitive. Example 2.10 yields an observation how Theorem 1.1 holds.

Example 2.10. Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be defined by forbidden set $\{01\}$. Apparently, Ω is extensible. We give the equivalence between the extensibility of Ω and the transitivity of $\mathsf{X}_\Omega^{(\ell)}$ a brief examination through the following discussions with two ℓ's.

1. Given that Ω is extensible. Suppose $\ell = 4$. Let $u' = 111111$ and $v' = 1111$ be two words in $\mathsf{X}_\Omega^{(\ell)}$. Consider $\alpha = 7$, a prime number greater than $\ell = 4$ and $\xi(u') = 6$. It follows from the extensibility of Ω that for any $k \in \mathbb{N}_0$ there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} x)|_{s(v')} = v'$.

2. Suppose $\ell = 2$ and $\mathsf{X}_\Omega^{(\ell)}$ is transitive. To show that $u = 101 \in B(\Omega)$ is extensible at position 5, pick $\alpha = 1, k = 2$, $u' = 0^{32}$, and $v' = 1001$, there exists $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} y)|_{s(v')} = v'$. Let $x := (y_{\alpha\ell^{(k+i)}})_{i\in\mathbb{N}}$. Then $x_{[5,7]} = 101$.

The following example provides an intuitive viewpoint for examining Theorem 1.2.

Example 2.11. For each $P \subset \mathbb{N}_0$ with $0 \in P$, the *spacing shift* introduced in [12] is defined as

$$\Sigma_P = \{s \in \{0,1\}^{\mathbb{N}} : s_i = s_j = 1 \Rightarrow |j - i| \in P\}. \tag{4}$$

Let $P = \mathbb{N}_0 \setminus \{2 + 10^N : N \in \mathbb{N}_0\}$ and $\Omega = \Sigma_P$. Then Ω is weakly mixing but not mixing since P is thick (see [4] for more details). We use the following examples to show that **1.** Ω is weakly mixing if $\mathsf{X}_\Omega^{(\ell)}$ is ℓ-directional mixing; **2.** $\mathsf{X}_\Omega^{(\ell)}$ is ℓ-directional mixing if Ω is weakly mixing.

1. Let $\ell = 2$ and $u_1 = 11, u_2 = 111, v_1 = 111, v_2 = 11 \in B(\Omega)$. It is easily seen that $u' = 1110010000010000, v' = 111101 \in B(\mathsf{X}_\Omega^{(\ell)})$ and $u'|_{\Lambda_1} = u_1 000$, $u'|_{\Lambda_3} = u_2$,

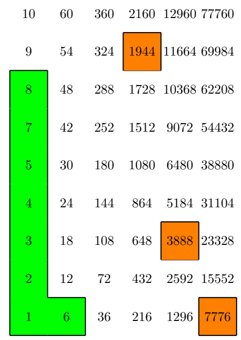
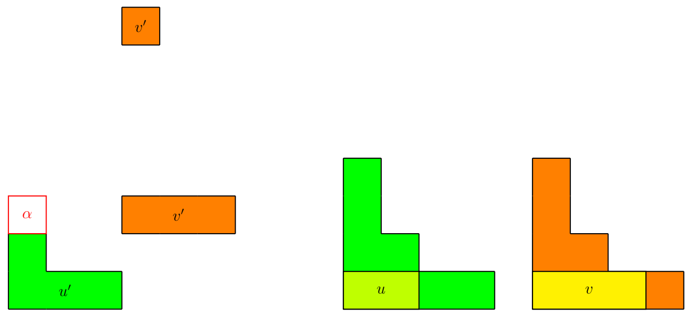


FIGURE 4. The figure gives an graphical representation of Example 2.12. Two words $u' = 1^8, v' = 1^5 \in B(\mathsf{X}_\Omega^{(6)})$ can be concatenated when $\alpha\ell^k = 243$, where the support of u' and $(\Pi_{|u'|\alpha\ell^k} x)|_{s(v')}$ are colored in green and orange respectively.

$v'|_{\Lambda_1} = v_1$, and $v'|_{\Lambda_3} = v_2$. More specifically, with $\alpha = 1$ and $k = 4$, there is an $x \in \mathsf{X}_\Omega^{(\ell)}$ given that $x|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} x)|_{s(v')} = (\Pi_{\ell^8} x)|_{s(v')} = v'$ and $x_i = 0$ otherwise. Therefore, u_1, v_1 are connected in $x|_{\Lambda_1}$ and u_2, v_2 are connected in $x|_{\Lambda_3}$.
2. Let $\ell = 4$ and $u' = 111111, v' = 1111 \in B(\mathsf{X}_\Omega^{(\ell)})$. Then $|u'| = \alpha_1\ell^{k_1} = 6 \cdot 4^0, |v'| = 4$, and $\alpha_1|v'| = 24$. It follows from Corollary 3.2 that there exists $M = 2$ such that for every $\alpha \in A_\ell, k \in \mathbb{N}_0$, and $i \in A_{\ell,|v'|}$, $i|u'|\alpha\ell^k = j\ell^{k_1+k+c}$ for some $j \in A_\ell$ and $c \leq M$. To connect $u'|_{\Lambda_i}$ and $v'|_{\Lambda_j}$ for $i \in A_{\ell,|u'|}$ and $j \in A_{\ell,|v'|}$, let $k = 3$ be fixed so that $[k, k + c + \max_i |u'|_{\Lambda_i}| + \max_i |v'|_{\Lambda_i}|] \subset [4, 10] \subset P$. Then for each $\alpha \in A_4$ there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_{s(u')} = u'$ and $(\Pi_{4^4\alpha|u'|} x)|_{s(v')} = v'$ and $x_i = 0$ elsewhere. For the case where $\alpha = 1$, the supports of u' and $x|_{4^4|u'|s(v')}$ are colored in green and orange respectively in Figure 3.

Next, we use the following example to verify Theorem 1.3.

Example 2.12. Let $P = \{0, 1\} \cup [3, \infty)$ and $\Omega = \Sigma_P$. Then Ω is mixing since P is cofinite (cf. [4]). In this case, given $u, v \in B(\Omega)$, if $x \in \mathcal{A}^{\mathbb{N}}$ satisfying $x_i = 0$ except $x|_{[1,|u|]} = u$ and $x|_{[|u|+3+1,|u|+3+|v|]} = v$, then $x \in \Omega$. We verify directly that **1.** $\mathsf{X}_\Omega^{(\ell)}$ is mixing if Ω is mixing and that **2.** Ω is mixing if $\mathsf{X}_\Omega^{(\ell)}$ is mixing.
1. Suppose $\ell = 6$, $u' = 1^8$ and $v' = 1^5$. Then, for every $\alpha\ell^k \geq 6^3 = 216$, there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} x)|_{s(v')} = v'$. For example, when $\alpha\ell^k = 243$ the support of u' and v' in x are colored in green and orange respectively in Figure 4. It is seen there that $s(u')$ and $s(v')$ are indeed separated at a distance greater than or equal to 3.
2. Let $\ell = 2$. Let $u = 1001$ and $v = 011$. It is possible to find $u' := 10000001 \in B_{\ell^{|u|-1}}(\mathsf{X}_\Omega^{(\ell)})$ and $v' := 0101 \in B_{\ell^{|v|-1}}(\mathsf{X}_\Omega^{(\ell)})$ such that $u'|_{\Lambda_1} = u$ and $v'|_{\Lambda_1} = v$. Then, for every $\alpha\ell^k \geq \ell^3$ there is $y^{(\alpha\ell^k)} \in \mathsf{X}_\Omega^{(\ell)}$ such that $y^{(\alpha\ell^k)}|_{s(u')} = u'$ and $(\prod_{|u'|\alpha\ell^k} y^{(\alpha\ell^k)})|_{s(v')} = v'$. In particular, for every $m \geq 3$, the configuration $x := y^{(\ell^m)}|_{\Lambda_1} \in \Omega$ satisfies that $x_{[1,|u|]} = u$ and $x_{[m+1,m+|v|]} = v$.



(A) (B)

FIGURE 5. An illustration for the equivalence between extensibility of Ω and transitivity of $\mathsf{X}_\Omega^{(\ell)}$. (A) Whenever Ω is extensible, the transitivity of $\mathsf{X}_\Omega^{(\ell)}$ can be derived by separate the given words. (B) If $\mathsf{X}_\Omega^{(\ell)}$ is transitive, then Ω is extensible.

Notably, none of above relations are equivalent to transitivity of Ω. One may refers to Example 2.9 to see that it is not equivalent to Theorem 1.1. As for Theorem 1.2, an example is given as follows.

Example 2.13. Let $\Omega \subset \{0,1\}^{\mathbb{N}}$ be defined by forbidden set $\{00, 11\}$. Then, Ω is transitive yet neither totally transitive nor weakly mixing, and $\mathsf{X}_\Omega^{(\ell)}$ satisfies all properties in Theorem 1.1. However, it is not ℓ-directional mixing. It can be verified by considering patterns $u' = 0110$, $v' = 1011$ in $\mathsf{X}_\Omega^{(2)}$, and $\alpha = 1$. If such $y \in \mathsf{X}_\Omega^{(2)}$ exists, then k is required to be even for $u'|_{\Lambda_1}$ and $v'|_{\Lambda_1}$, and odd for $u'|_{\Lambda_3}$ and $v'|_{\Lambda_3}$. This contradicts the existence of k. Nevertheless, it is consistent with Theorem 1.2 and with Theorem 1.1.

3. Proofs of Main Theorems

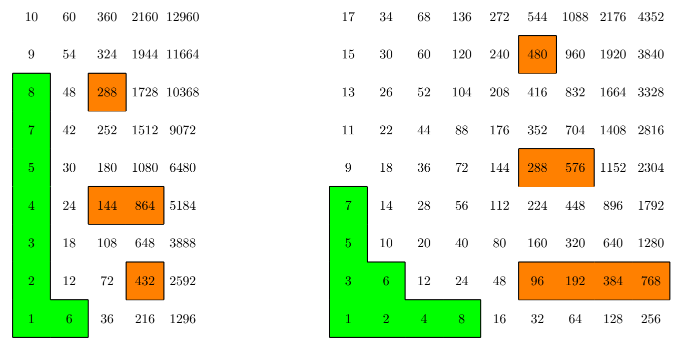
This section is devoted to demonstrating the main theorems of this paper. We start from the equivalence between extensibility of Ω and transitivity of $\mathsf{X}_\Omega^{(\ell)}$.

Proof of Theorem 1.1. The theorem is proved in the order (3) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1). The idea of the proof is provided in Figure 5.

(3) $\Rightarrow$ (2). It follows directly from definition.

(2) $\Rightarrow$ (1). We prove that for $u \in B(\Omega)$ and $m \in \mathbb{N}_0$ there is $x \in \Omega$ such that $x_{[m+1,m+|u|]} = u$. Let $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ such that $|u'| = \ell^m$, $|v'| = \ell^{|u|-1}$ and $v'|_{\Lambda_1} = u$. By transitivity of $\mathsf{X}_\Omega^{(\ell)}$, there are $\alpha \in A_\ell$, $k \in \mathbb{N}_0$ and $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} y)|_{s(v')} = v'$. The proof is completed by letting $x_i := (y_{\alpha\ell^{k+i-1}})$.

(1) $\Rightarrow$ (3). We prove that for arbitrary blocks $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$, and for $k \in \mathbb{N}_0$, there exists $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} y)|_{s(v')} = v'$ whenever $\alpha \in$



(A) $u \in B_8(\mathsf{X}_\Omega^{(6)})$ (B) $u \in B_8(\mathsf{X}_\Omega^{(2)})$

FIGURE 6. Suppose $x \in \mathsf{X}_\Omega^{(2)}$ and $u \in B_8(\mathsf{X}_\Omega^{(2)})$. (A) $(x|_{\Lambda_i \cap 144 s(u)})$'s are left-aligned since 6 is not a prime. (B) $(x|_{\Lambda_i \cap 96 s(u)})$'s are left-aligned since 2 is a prime.

$\mathbb{P}\backslash\{\ell\}$ with $\alpha > \ell$. Note that for each $i \in A_{\ell,|u'|}$ and $j \in A_{\ell,|v'|}$, $(|u'|\alpha\ell^k\Lambda_j)\cap\Lambda_i = \varnothing$ always holds. In other words, $s(u')$ and $|u'|\alpha\ell^k s(v')$ are separated in the sense every Λ_i is intersected by at most one of the two sets. The existence of $y \in \mathsf{X}_\Omega^{(\ell)}$ is thus guaranteed by the extensibility of Ω on every Λ_i.

(1) $\Rightarrow$ (5). We claim that for $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ and for any $\alpha \in A_L$ there exists $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u', (\Pi_{|u'|\alpha L^k} y)|_{s(v')} = v'$ whenever $p^k > \xi(u)$, $p|L$ but $p \nmid \ell$. Since $p^k > \xi(u')$, it follows immediately that

$$|u'|\alpha L^k \mathbb{N} \cap \Lambda_i = \varnothing \quad \text{for} \quad 1 \le i \le \xi(u').$$

In other words, $|u'|\alpha L^k s(v')$ is a subset of $\mathbb{N} \setminus \cup_{1\le i \le \xi(u')}\Lambda_i$. Therefore, for each $i \in A_\ell$, there exists $x_i \in \Omega$ such that

$$\begin{cases} x_i|_{[1,|s(u'|_{\Lambda_i})|]} = u'|_{\Lambda_i}, & \text{if } 1 \le i \le \xi(u'); \\ x_i|_{\log_\ell \frac{|u'|\alpha L^k i}{i'} + [1,|v'|_{\Lambda_i}|]} = v'|_{\Lambda_i}, & \text{otherwise, where } |u'|\alpha L^k \Lambda_i \subset \Lambda_{i'}; \end{cases}$$

since Ω is extensible. Let $y \in \mathcal{A}^{\mathbb{N}}$ be defined by $y|_{\Lambda_i} = x_i$. Then $y \in \mathsf{X}_\Omega^{(\ell)}$ is the desired result.

(5) $\Rightarrow$ (4). It holds automatically since (4) is a particular case of (5).

(4) $\Rightarrow$ (1). We prove that for $u \in B(\Omega)$ and $m \in \mathbb{N}_0$ there is $x \in \Omega$ such that $x_{[m+1,m+|u|]} = u$. Let $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ such that $|u'| = \ell^{m+1}$, that $|v'| = \ell^{|u|-1}$ and that $v'|_{\Lambda_1} = u$. By L-directional mixing property of $\mathsf{X}_\Omega^{(\ell)}$, there is $k \in \mathbb{N}_0$ and $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|L^k} y)|_{s(v')} = v'$. Suppose that $|u'|L^k = i\ell^{c+m+1}$ for some $c \in \mathbb{N}_0$, i.e., $|u'|L^k \in \Lambda_i$ for some $\ell \nmid i$. The proof is complete by letting $x := (y_{i\ell^{c+j}})_{j\in\mathbb{N}}$. $\square$

Next, we link the weakly mixing property of Ω and the ℓ-directional mixing property of $\mathsf{X}_\Omega^{(\ell)}$. Proposition 2.6 implies that the multiplicative transformation breaks the topological structure of pattern even more significantly in the case ℓ

is not a prime than the case ℓ is a prime. More precisely, given $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ and $k \in \mathbb{N}_0$, for every $\alpha \in A_\ell$ and every $i \in A_{\ell,|v'|}$, the product $(|u'|\alpha\ell^k)i$ can be represented as $j_{\alpha,i}\ell^{k_1+k+c_{\alpha,i}}$ for some $\ell \nmid j_{\alpha,i}$, where $|u'| = \alpha_1\ell^{k_1}$ and $c_{\alpha,i}$ is the extra "offset" introduced by the product $\alpha_1\alpha i = j_{\alpha,i}\ell^{c_{\alpha,i}}$. Hence, $(|u'|\alpha\ell^k)s(v'|_{\Lambda_i})$ is not "left-aligned" (see Figure 6). Nevertheless, Lemma 3.1 and Corollary 3.2 show that these offsets $\{c_{\alpha,i}\}_{\alpha\in A_\ell, i\in A_{\ell,|v'|}}$ are bounded.

Lemma 3.1. *Given any $N \in \mathbb{N}$, there exists $M \in \mathbb{N}$ so that $A_\ell A_{\ell,N} := \{ab : a \in A_\ell, b \in A_{\ell,N}\} \subset \cup_{i=0}^M \ell^i A_\ell$.*

Proof. By the fundamental theorem of arithmetic, there exists the unique prime factorization of ℓ as $\ell = p_1^{m_1} p_2^{m_2} \dots p_r^{m_r}$. Let

$$M = \max\{\lfloor \frac{k_{p_i}}{m_i} \rfloor : 1 \le i \le m, k \in A_{\ell,N}\} + 1,$$

where $k_{p_i} = \max\{m \in \mathbb{N}_0 : p_i^m | k\}$. Given $a \in A_\ell$ and $b \in A_{\ell,N}$, there exists $1 \le i \le r$ such that $\lfloor \frac{a_{p_i}}{m_i} \rfloor = 0$. Since $\lfloor \frac{b_{p_i}}{m_i} \rfloor \le M - 1$, it comes immediately that $\lfloor \frac{(ab)_{p_i}}{m_i} \rfloor \le M$. Hence, $ab \in \cup_{i=0}^M \ell^i A_\ell$ and the proof is complete. □

Corollary 3.2. *Suppose $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ with $|u'| = \alpha_1\ell^{k_1}$. There exists $M \in \mathbb{N}$ such that for every $\alpha \in A_\ell, k \in \mathbb{N}_0$, and $i \in A_{\ell,|v'|}$, $i|u'|\alpha\ell^k = j\ell^{k_1+k+c}$ for some $j \in A_\ell$ and $c \le M$.*

Proof. Observe that there exists $M_1 \in \mathbb{N}$ such that $\alpha_1 A_{\ell,|v'|} \subseteq \cup_{i=0}^{M_1} \ell^i A_{\ell,\alpha_1|v'|}$. Lemma 3.1 shows that there exists $M_2 \in \mathbb{N}$ so that $A_\ell A_{\ell,\alpha_1|v'|} \subset \cup_{i=0}^{M_2} \ell^i A_\ell$. It follows that

$$|u'|A_\ell \ell^k A_{\ell,|v'|} \subseteq \ell^{k+k_1} A_\ell \cup_{i=0}^{M_1} \ell^i A_{\ell,\alpha_1|v'|} \subseteq \cup_{i=0}^{M_1+M_2} \ell^{k+k_1+i} A_\ell,$$

the desired result follows by letting $M = M_1 + M_2$. □

With the estimation of the offsets, we are ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. The proof is divided into three parts. First we show that (1) ⇔ (2). After demonstrating the equivalence of (1), (3), and (4), it follows that (1) ⇔ (5).

(1) ⇒ (2). Given $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ with $|u'| = \alpha_1\ell^{k_1}$ for some $\alpha_1 \in A_\ell$, Corollary 3.2 indicates there exists $M \in \mathbb{N}$ such that for $\alpha \in A_\ell, k \in \mathbb{N}_0$, and $i \in A_{\ell,|v'|}$,

$$i|u'|\alpha\ell^k = i'\ell^{k_1+k+c_{i,\alpha,\alpha_1}} \quad \text{for some } i' \in A_\ell \text{ and } c_{i,\alpha,\alpha_1} \le M.$$

Let

$$\Delta = \{(u'|_{\Lambda_i}, w^{(c,j)}(v'|_{\Lambda_j})) : 1 \le i \le \xi(u'), 1 \le j \le \xi(v'), \\ 0 \le c \le M\},$$

be a finite collection of pairs of blocks in Ω, where ϵ is the empty word and $w^{(r,j)} \in B(\Omega) \cup \{\epsilon\}$ is chosen so that $w^{(c,j)}(v'|_{\Lambda_j}) \in B(\Omega)$. Since $\mathsf{X}_\Omega^{(\ell)}$ is weakly mixing, there exists $K \in \mathbb{N}$ such that for $(\overline{u}, \overline{v}) \in \Delta$, there exists $\overline{w} \in B_{K-|\overline{u}|}(\Omega)$ such that

$\overline{uwv} \in B(\Omega)$ by Proposition 2.3. Note that $K \geq |u|_{\Lambda_i}|$ for every $i \in A_{\ell,|u'|}$ and so $K \geq k_1$.

Next we show that for each $\alpha \in A_\ell$ there exists $x \in \mathsf{X}_\Omega^{(\ell)}$ such that $x|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^{K-k_1}} x)|_{s(v')} = v'$. Observe that $(\Pi_{|u'|\alpha\ell^{K-k_1}} x)|_{s(v')} = v'$ if and only if

$$x_{j'}|_{\log_\ell \frac{|u'|\alpha\ell^{K-k_1}j}{j'}+\left[1,|v'|_{\Lambda_j}|\right]} = v'|_{\Lambda_j} \quad \text{for} \quad j \in A_{\ell,|v'|},$$

where $j' \in A_\ell$ satisfies $(|u'|\alpha\ell^{K-k_1})\Lambda_j \subset \Lambda_{j'}$ and $x_{j'} := x|_{\Lambda_{j'}}$. In fact, for every $j \in A_{\ell,|u'|}$, there exists at most one $j' \in A_{\ell,|v'|}$ such that $|u'|\alpha\ell^{K-k_1}\Lambda_j \subseteq \Lambda_i$. The construction of x is as follows. For $i \in A_{\ell,|u'|}$, if there exists $j \in A_{\ell,|v'|}$ such that $|u'|\alpha\ell^{K-k_1}\Lambda_j = \ell^{K+c_{i,\alpha,\alpha_1}}\Lambda_i \subseteq \Lambda_i$, the discussion in the previous paragraph implies the existence of word $(u'|_{\Lambda_i})\overline{w}w^{(c_{i,\alpha,\alpha_1},j)}(v'|_{\Lambda_j}) \in B_{K-|\overline{u}|}(\Omega)$ for some $\overline{w} \in B_{K-|\overline{u}|}(\Omega)$, and thus implies the existence of x_i such that

$$x_i|_{s(u'|_{\Lambda_i})} = u'|_{\Lambda_i} \text{ and } x_i|_{\log_\ell \frac{|u'|\alpha\ell^{K-k_1}j}{i}+\left[1,|v'|_{\Lambda_j}|\right]} = v'|_{\Lambda_j},$$

which also means that $x_i|_{\log_\ell \frac{|u'|\alpha\ell^{K-k_1}j}{i}+m} = v'_{j\ell^{m-1}}$ for $m \in [1, |v'|_{\Lambda_j}|]$; otherwise, the existence of x_i comes from the extensibility of Ω. For the case where $i \in A_\ell \setminus A_{\ell,|v'|}$, the existence of x_i also comes from the extensibility of Ω. The desired $x \in \mathsf{X}_\Omega^{(\ell)}$ then follows by letting $x|_{\Lambda_i} = x_i$.

(2) $\Rightarrow$ (1). To show that for $u_1, u_2, v_1, v_2 \in B(\Omega)$ there are $x_1, x_2 \in \Omega$ and $k \in \mathbb{N}$ such that $x_i|_{s(u_i)} = u_i$ and that $x_i|_{k+s(v_i)} = v_i$ for $i = 1, 2$, let $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ such that u_1, u_2, v_1, v_2 are subword of $u'|_{\Lambda_1}, u'|_{\Lambda_{\ell+1}}, v'|_{\Lambda_1}, v'|_{\Lambda_{\ell+1}}$ respectively, and that $|u'| = \ell^{k_1}$ for some $k_1 \in \mathbb{N}$. Since $\mathsf{X}_\Omega^{(\ell)}$ is ℓ-directional mixing, there is a $k \in \mathbb{N}$ and an $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|\ell^k} y)|_{s(v')} = v'$. The proof is completed by letting $x_1 = y|_{\Lambda_1}$ and $x_2 = y|_{\Lambda_{l+1}}$.

The discussion of (1) $\Rightarrow$ (3) and (4) $\Rightarrow$ (1) are similar to that of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1), respectively. Since (4) is a special case of (3), the equivalence of (1), (3), and (4) then follows.

The demonstration of (1) $\Leftrightarrow$ (5) is analogous to the derivation of (1) $\Leftrightarrow$ (3) above together with the proof of Theorem 1.1 (1) $\Leftrightarrow$ Theorem 1.1 (5). Thus the detailed elucidation is omitted for the sake of compactness. □

We finish this section with the proof that Ω is mixing if and only if $\mathsf{X}_\Omega^{(\ell)}$ is mixing.

Proof of Theorem 1.3. The theorem is proved in the order (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2). Suppose $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ are given. Since Ω is mixing, there exists $N_0 \in \mathbb{N}$ such that for all $m \geq N_0$, $i \in A_{\ell,|u'|}, j \in A_{\ell,|v'|}$ there exists $x = x(i, j, m) \in \Omega$ such that $x|_{[1,|u'|_{\Lambda_i}|]} = u'|_{\Lambda_i}, x|_{[m+|u'|_{\Lambda_i}|+1,m+|u'|_{\Lambda_i}|+|v'|_{\Lambda_j}|]} = v'|_{\Lambda_j}$. We claim that for $\alpha \in \mathbb{N}, k \in \mathbb{N}_0$ such that $\alpha\ell^k \geq \ell^{N_0}$ there is $y \in \mathsf{X}_\Omega^{(\ell)}$ satisfying $y|_{s(u')} = u'$ and $(\Pi_{|u'|(\alpha\ell^k)} y)|_{s(v')} = v'$, which is equivalent to mixing property in $\mathsf{X}_\Omega^{(\ell)}$ by Proposition 2.5. Similar to the proof of Theorem 1.2, it suffices to show that whenever $(|u'|\alpha\ell^k)\Lambda_j \subset \Lambda_i$ for some $1 \leq i \leq \xi(u')$ and $1 \leq j \leq \xi(v')$, there exists

Ω \ $\mathsf{X}_\Omega^{(\ell)}$	transitivity	ℓ-directional mixing	mixing
extensibility	EQ	T	T
transitivity	F	T	T
weakly mixing	F	EQ	T
mixing	F	F	EQ

TABLE 1. Summary of the main results. In this table, 'T' means that the property in Ω implies the property in $\mathsf{X}_\Omega^{(\ell)}$ and 'F' means the opposite, and 'EQ' means two properties are equivalent.

$x_i \in \Omega$ such that

$$x_i|_{[1,|u'|_{\Lambda_i}|]} = u'|_{\Lambda_i} \quad \text{and} \quad x_i|_{\log_\ell \frac{|u'|\alpha\ell^k j}{i} + [1,|v'|_{\Lambda_j}|]} = v'|_{\Lambda_j}.$$

Equivalently, we need to show that $\log_\ell \frac{L_2}{i} - \log_\ell \frac{L_1}{i} \geq N_0$, where $L_1 = \max(s(u') \cap \Lambda_i)$ and $L_2 = \min(|u'|\alpha\ell^k(s(v') \cap \Lambda_j))$. Indeed,

$$\log_\ell \frac{L_2}{i} - \log_\ell \frac{L_1}{i} = \log_\ell \frac{|u'|}{L_1}\alpha\ell^k j \geq N_0.$$

Therefore, Ω being mixing implies that $\mathsf{X}_\Omega^{(\ell)}$ is mixing.

(2) $\Rightarrow$ (3). This could be proved by choosing proper α or k in (2).

(3) $\Rightarrow$ (1). Given $u, v \in B(\Omega)$, let $u', v' \in B(\mathsf{X}_\Omega^{(\ell)})$ such that $|u'| = \ell^{|u|-1}, |v'| = \ell^{|v|-1}$, $u'|_{\Lambda_1} = u$, and $v'|_{\Lambda_1} = v$. Hence, there exists $N \in \mathbb{N}$ such that for $\alpha \in A_\ell, k \geq N$ there exists $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{[1,|u'|]} = u'$ and $(\Pi_{|u'|\alpha\ell^k} y)|_{[1,|v'|]} = v'$. We claim that for $m \geq N$ there exists $x \in \Omega$ such that $x|_{[1,|u|]} = u, x|_{[|u|+m+1,|u|+m+|v|]} = v$. Indeed, let $\alpha = 1$ and $k = m+1 > N$, there exists $y \in \mathsf{X}_\Omega^{(\ell)}$ such that $y|_{s(u')} = u'$ and $(\Pi_{|u'|\alpha\ell^k} y)|_{s(v')} = v'$. In other words, $(y|_{\Lambda_1})|_{[1,|u|]} = u$ and

$$(\Pi_{|u'|\alpha\ell^k} y)_i = y_{i\ell^{|u|+m}} = v_j \quad \text{for} \quad i = \ell^{j-1}, j = 1, \ldots, |v|.$$

The proof is then complete by letting $x = y|_{\Lambda_1}$. $\square$

4. Summary and Discussion

Suppose Ω is a traditional shift space and $\mathsf{X}_\Omega^{(\ell)}$ is the corresponding multiplicative shift space for some $\ell > 1$. We investigate the relations between the mixing properties of Ω and $\mathsf{X}_\Omega^{(\ell)}$. After introducing the ℓ-directional mixing property, we reveal some if-and-only-if connection between mixing properties of two systems. Table 1 summarizes the main results of this paper. It is seen that there are still open problems remained to be studied. We list these problems of interest in the following, some of which are in preparation.

Question 4.1. In Theorem 1.2, (5) is equivalent to the others if ℓ is a prime number. Does this hold for arbitrary $\ell > 1$?

Question 4.2. Is there any equivalent condition for $\mathsf{X}_\Omega^{(\ell)}$ as transitivity of Ω?

Question 4.3. Do Theorems 1.1, 1.2, and 1.3 hold for two-sided multiplicative subshift $\mathsf{X}_{\Omega}^{(\ell)} \subset \mathcal{A}^{\mathbb{Z}}$?

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