BACKGROUNDS IN LARGE DEVIATION PRINCIPLES

YU-LIANG WU

Table of Contents

1. Existence and Uniqueness of the LDP

We begin by defining the δ-rate function, which is useful for comparing rate functions.

Definition 1.1. The *δ-rate function* I^δ associated with a rate function I is defined as

$$I^\delta(x) = \min\left\{ I(x) - \delta, \frac{1}{\delta} \right\}. \tag{1.1}$$

Proposition 1.2 (Uniqueness)**.** *A family of probability measures $\{\mu_\varepsilon\}$ on a regular topological space can have at most one associated rate function.*

Proof. Suppose I_1 and I_2 are two rate functions associated with the LDP of μ_ε, and assume there exists a point x_0 such that $I_1(x_0) < I_2(x_0)$. By lower semicontinuity, there exists an open neighborhood A of x_0 such that $\inf_{x\in A} I_2(x) \geq I_2^\delta(x_0)$, where I_2^δ is the δ-rate function defined in Definition 1.1. By regularity, we can further assume $\inf_{x\in\overline{A}} I_2(x) \geq I_2^\delta(x_0)$. Consequently,

$$-I_2^\delta(x_0) \geq \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geq \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A) \geq -I_1(x_0),$$

which leads to a contradiction as $\delta \to 0$. □

Remark 1.3.

(1) *All metric spaces are regular Hausdorff spaces.*

(2) *If $\mathcal{X}$ is regular and locally compact, the LDP condition in Proposition 1.2 can be relaxed to a weak LDP.*

Proposition 1.4 (Existence)**.** *Let $\mathcal{A}$ be a topological base for space $\mathcal{X}$. For every $A \in \mathcal{A}$, define*

$$\underline{\mathcal{L}}_A := \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A), \quad \overline{\mathcal{L}}_A := \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(A),$$

and

$$\underline{I}(x) := -\inf_{A\in\mathcal{A}: x\in A} \underline{\mathcal{L}}_A, \quad \overline{I}(x) := -\inf_{A\in\mathcal{A}: x\in A} \overline{\mathcal{L}}_A.$$

If $\underline{I} = \overline{I}$, then μ_ε satisfies the weak LDP with the rate function $\underline{I}$.

Proof. That $\underline{I}$ and $\overline{I}$ are indeed rate functions follows naturally from their definitions.

To establish the lower bound of the LDP, observe that for every open set G and any $x \in G$, there exists an $A \in \mathcal{A}$ such that $x \in A \subset G$. This implies

$$\underline{\mathcal{L}}_G \geq \underline{\mathcal{L}}_A \geq -\underline{I}(x),$$

from which the desired lower bound follows.

Regarding the upper bound, for any compact set F and every point $x \in F$, we can find a neighborhood A_x satisfying $\overline{\mathcal{L}}_{A_x} \leq -\overline{I}^{\delta}(x)$. Since F is compact, it admits a finite subcover $\{A_{x_i}\}_{i=1}^{m}$; thus,

$$\overline{\mathcal{L}}_F \leq \max_{1 \leq i \leq m} \overline{\mathcal{L}}_{A_{x_i}} \leq \max_{1 \leq i \leq m} -\overline{I}^{\delta}(x_i) \leq -\inf_{x \in F} \overline{I}^{\delta}(x).$$

Taking $\delta \to 0$ completes the proof. □

Remark 1.5. *Proposition 1.4 holds for a parametrized family* $\{\mu_{\varepsilon,\sigma}\}$ *for any given* $\sigma \in \Sigma$ *if we define:*

$$\underline{\mathcal{L}}_A := \liminf_{\varepsilon \to 0} \varepsilon \log \left[\inf_{\sigma \in \Sigma} \mu_{\varepsilon,\sigma}(A) \right], \quad \overline{\mathcal{L}}_A := \limsup_{\varepsilon \to 0} \varepsilon \log \left[\sup_{\sigma \in \Sigma} \mu_{\varepsilon,\sigma}(A) \right].$$

This is useful for describing processes such as Markov chains conditioned on the initial state σ.

We also have a partial converse of Proposition 1.4.

Proposition 1.6. *Suppose that* $\{\mu_\varepsilon\}$ *satisfies the LDP in a topological space* $\mathcal{X}$ *with rate function* I. *Then, for any topological base* $\mathcal{A}$ *of* $\mathcal{X}$, $I = \underline{I} = \overline{I}$, *where these functions are defined as in Proposition 1.4.*

Proof. Suppose $\underline{I}(x) > \overline{I}(x)$ for some x. By the lower semicontinuity of I and the regularity of $\mathcal{X}$, for every $\delta > 0$, there exists an open neighborhood A of x such that

$$-I^{\delta}(x) \geq -\inf_{y \in \overline{A}} I(y) \geq \overline{\mathcal{L}}_A \geq -\overline{I}(x).$$

Conversely, by the definition of the LDP, for every $\delta > 0$, there exists an open neighborhood A of x such that

$$-I(x) \leq -\inf_{y \in \mathring{A}} I(y) \leq \underline{\mathcal{L}}_A \leq -\underline{I}^{\delta}(x).$$

Combining these inequalities yields the desired result. □

References

Research Unit of Mathematical Sciences, University of Oulu, P.O. Box 3000, 90014
University of Oulu, Finland.
Email address: `yu-liang.wu@oulu.fi`
URL: `s92077.github.io`